MAT 142: Inverse Functions

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Reminders

At our last meeting, we discussed each of the following:

- Book functions and committing their shapes to memory.
- Vertical and horizontal shifts, stretches, compressions, and reflections.
- Combining transformations to build graphs of more general functions.
- Landmarks on graphs: roots, y-intercept, local maxima / minima, intervals of increase / decrease.

Try the following warm-up problems.

Problem 1: Consider $f(x) = \dfrac{x-5}{4}$ and $g(x) = 4x + 5$.

- Describe, in words, how the function f acts on its inputs.
- Describe, in words, how the function g acts on its inputs
- Find $f(g(x))$ and $g(f(x))$.

Problem 2: Consider $j(x) = 2x^3 + 8$ and $k(x) = \left(\dfrac{x-8}{2}\right)^{1/3}$.

- Describe, in words, how the function j acts on its inputs.
- Describe, in words, how the function k acts on its inputs
- Find $j(k(x))$ and $k(j(x))$.

What do you notice about each of these pairs?

Motivating Application

A function $f(c)$ converts temperatures from degrees Celsius to degrees Fahrenheit:

$$f(c) = \frac{9}{5}c + 32$$

Find a function that converts temperatures from degrees Fahrenheit back to degrees Celsius.

Before we work through this, consider the following:

- What does $f(0) = 32$ tell you? Does it give you any information about a function that converts Fahrenheit back to Celsius?
- What does the solution to $f(c) = 40$ mean?

We'll return to this application at the end of today's class.

Objectives

After today's class meeting, you should be able to:

- Define what it means for two functions to be inverses of one another.
- Use composition to determine whether two functions are inverses.
- Identify which functions have inverses using the horizontal line test.
- Use the *swap-and-solve* method to find the inverse of an invertible function.

What is an Inverse?

Definition (Inverse Functions): The functions $f(x)$ and $g(x)$ are *inverses* of one another if

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x$$

In intuitive terms, f and g are inverses if f *undoes* whatever g *does* to its inputs, and vice-versa.

Think of it this way:

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow x$$

We write the inverse of f as f^{-1}.

Overloaded Notation Warning

$f^{-1}(x)$ means *the inverse of f*, not $\dfrac{1}{f(x)}$. The superscript -1 is notation for "inverse", not an exponent. You'll need to use context to infer the meaning of the "-1" superscript.

Completed Example 1

Problem: Determine whether $f(x) = 2x + 10$ and $g(x) = \frac{1}{2}x - 10$ are inverses of one another.

Solution. We check both compositions.

$$f(g(x)) = f\left(\frac{1}{2}x - 10\right)$$

Completed Example 1

Problem: Determine whether $f(x) = 2x + 10$ and $g(x) = \frac{1}{2}x - 10$ are inverses of one another.

Solution. We check both compositions.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{2}x - 10\right) \\ &= 2\left(\frac{1}{2}x - 10\right) + 10 \end{aligned}$$

Completed Example 1

Problem: Determine whether $f(x) = 2x + 10$ and $g(x) = \frac{1}{2}x - 10$ are inverses of one another.

Solution. We check both compositions.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{2}x - 10\right) \\ &= 2\left(\frac{1}{2}x - 10\right) + 10 \\ &= (x - 20) + 10 \end{aligned}$$

Completed Example 1

Problem: Determine whether $f\left(x\right) = 2x + 10$ and $g\left(x\right) = \dfrac{1}{2}x - 10$ are inverses of one another.

Solution. We check both compositions.

$$\begin{aligned} f\left(g\left(x\right)\right) &= f\left(\frac{1}{2}x - 10\right) \\ &= 2\left(\frac{1}{2}x - 10\right) + 10 \\ &= \left(x - 20\right) + 10 \\ &= x - 10 \end{aligned}$$

Completed Example 1

Problem: Determine whether $f(x) = 2x + 10$ and $g(x) = \frac{1}{2}x - 10$ are inverses of one another.

Solution. We check both compositions.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{2}x - 10\right) \\ &= 2\left(\frac{1}{2}x - 10\right) + 10 \\ &= (x - 20) + 10 \\ &= x - 10 \\ &\neq x \end{aligned}$$

Since $f(g(x)) \neq x$, we conclude that f and g are **not** inverses of one another. There is no need to check the other composition.

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

$$f(g(x)) = f\left(\frac{5}{x-1}\right)$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

$$\begin{aligned} f(g(x)) &= f\left(\frac{5}{x-1}\right) \\ &= \frac{\left(\frac{5}{x-1}\right)+5}{\left(\frac{5}{x-1}\right)} \end{aligned}$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

$$\begin{aligned}
f(g(x)) &= f\left(\frac{5}{x-1}\right) \\
&= \frac{\left(\frac{5}{x-1}\right)+5}{\left(\frac{5}{x-1}\right)} \\
&= \frac{\frac{5}{x-1}+\frac{5(x-1)}{x-1}}{\frac{5}{x-1}}
\end{aligned}$$

Completed Example 2

Problem: Determine whether $f\left(x\right) = \dfrac{x+5}{x}$ and $g\left(x\right) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f\left(g\left(x\right)\right)$ first.

$$\begin{aligned}
f\left(g\left(x\right)\right) &= f\left(\frac{5}{x-1}\right) \\
&= \frac{\left(\frac{5}{x-1}\right)+5}{\left(\frac{5}{x-1}\right)} \\
&= \frac{\frac{5}{x-1}+\frac{5(x-1)}{x-1}}{\frac{5}{x-1}} \\
&= \frac{\frac{5+5x-5}{x-1}}{\frac{5}{x-1}}
\end{aligned}$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

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&= \frac{\frac{5+5x-5}{x-1}}{\frac{5}{x-1}} \\
&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}}
\end{aligned}$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

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&= \frac{\frac{5+5x-5}{x-1}}{\frac{5}{x-1}} \\
&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}} \\
&= \left(\frac{5x}{x-1}\right)\cdot\left(\frac{x-1}{5}\right)
\end{aligned}
$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

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&= \frac{\frac{5+5x-5}{x-1}}{\frac{5}{x-1}} \\
&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}} \\
&= \left(\frac{5x}{x-1}\right)\cdot\left(\frac{x-1}{5}\right) \\
&= x \checkmark
\end{aligned}$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

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&= x \checkmark
\end{aligned}$$

Now we check $g(f(x))$.

Completed Example 2

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&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}} \\
&= \left(\frac{5x}{x-1}\right)\cdot\left(\frac{x-1}{5}\right) \\
&= x \checkmark
\end{aligned}
$$

Now we check $g(f(x))$.

$$
g(f(x)) = g\left(\frac{x+5}{x}\right)
$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

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f(g(x)) &= f\left(\frac{5}{x-1}\right) \\
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&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}} \\
&= \left(\frac{5x}{x-1}\right)\cdot\left(\frac{x-1}{5}\right) \\
&= x \checkmark
\end{aligned}$$

Now we check $g(f(x))$.

$$\begin{aligned}
g(f(x)) &= g\left(\frac{x+5}{x}\right) \\
&= \frac{5}{\left(\frac{x+5}{x}\right)-1}
\end{aligned}$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

Solution. We check $f(g(x))$ first.

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f(g(x)) &= f\left(\frac{5}{x-1}\right) \\
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&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}} \\
&= \left(\frac{5x}{x-1}\right)\cdot\left(\frac{x-1}{5}\right) \\
&= x \checkmark
\end{aligned}$$

Now we check $g(f(x))$.

$$\begin{aligned}
g(f(x)) &= g\left(\frac{x+5}{x}\right) \\
&= \frac{5}{\left(\frac{x+5}{x}\right)-1} \\
&= \frac{5}{\frac{x+5}{x}-\frac{x}{x}}
\end{aligned}$$

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Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

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f(g(x)) &= f\left(\frac{5}{x-1}\right) \\
&= \frac{\left(\frac{5}{x-1}\right)+5}{\left(\frac{5}{x-1}\right)} \\
&= \frac{\frac{5}{x-1} + \frac{5(x-1)}{x-1}}{\frac{5}{x-1}} \\
&= \frac{\frac{5+5x-5}{x-1}}{\frac{5}{x-1}} \\
&= \frac{\frac{5x}{x-1}}{\frac{5}{x-1}} \\
&= \left(\frac{5x}{x-1}\right) \cdot \left(\frac{x-1}{5}\right) \\
&= x \checkmark
\end{aligned}
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Now we check $g(f(x))$.

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\begin{aligned}
g(f(x)) &= g\left(\frac{x+5}{x}\right) \\
&= \frac{5}{\left(\frac{x+5}{x}\right)-1} \\
&= \frac{5}{\frac{x+5}{x} - \frac{x}{x}} \\
&= \frac{5}{\frac{x+5-x}{x}}
\end{aligned}
$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

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\end{aligned}$$

Now we check $g(f(x))$.

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g(f(x)) &= g\left(\frac{x+5}{x}\right) \\
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&= \frac{5}{\frac{x+5}{x}-\frac{x}{x}} \\
&= \frac{5}{\frac{x+5-x}{x}} \\
&= \frac{5}{\frac{5}{x}} \\
&= 5\cdot\left(\frac{x}{5}\right)
\end{aligned}$$

Completed Example 2

Problem: Determine whether $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{5}{x-1}$ are inverses of one another.

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&= \frac{5}{\frac{x+5}{x}-\frac{x}{x}} \\
&= \frac{5}{\frac{x+5-x}{x}} \\
&= \frac{5}{\frac{5}{x}} \\
&= 5\cdot\left(\frac{x}{5}\right) \\
&= x \checkmark
\end{aligned}$$

Since $f(g(x)) = x$ and $g(f(x)) = x$, we have $g = f^{-1}$.

Verifying Inverses Practice

Try It! 1: Show that $f\left(x\right)=2x+7$ and $g\left(x\right)=\dfrac{x-7}{2}$ are inverses of one another.

Try It! 2: Show that $f\left(x\right)=\dfrac{8x+2}{5}$ and $g\left(x\right)=\dfrac{5x-2}{8}$ are inverses of one another.

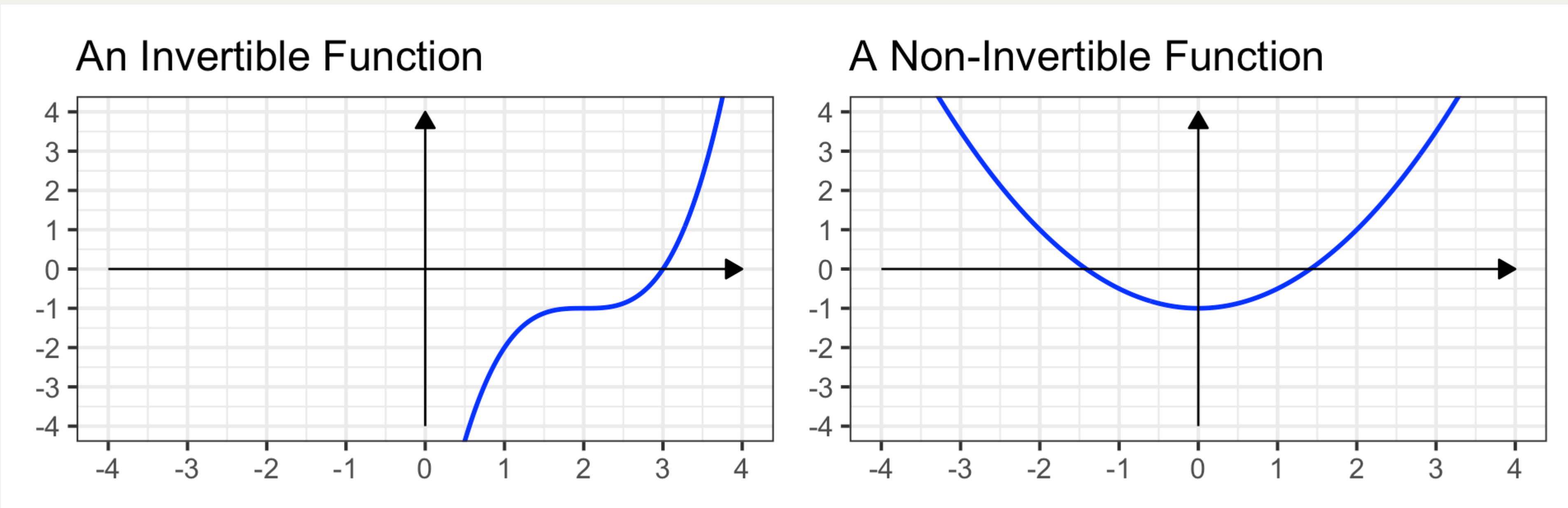
Reminder: You need to verify both $f\left(g\left(x\right)\right)=x$ and $g\left(f\left(x\right)\right)=x$.

Which Functions Have Inverses?

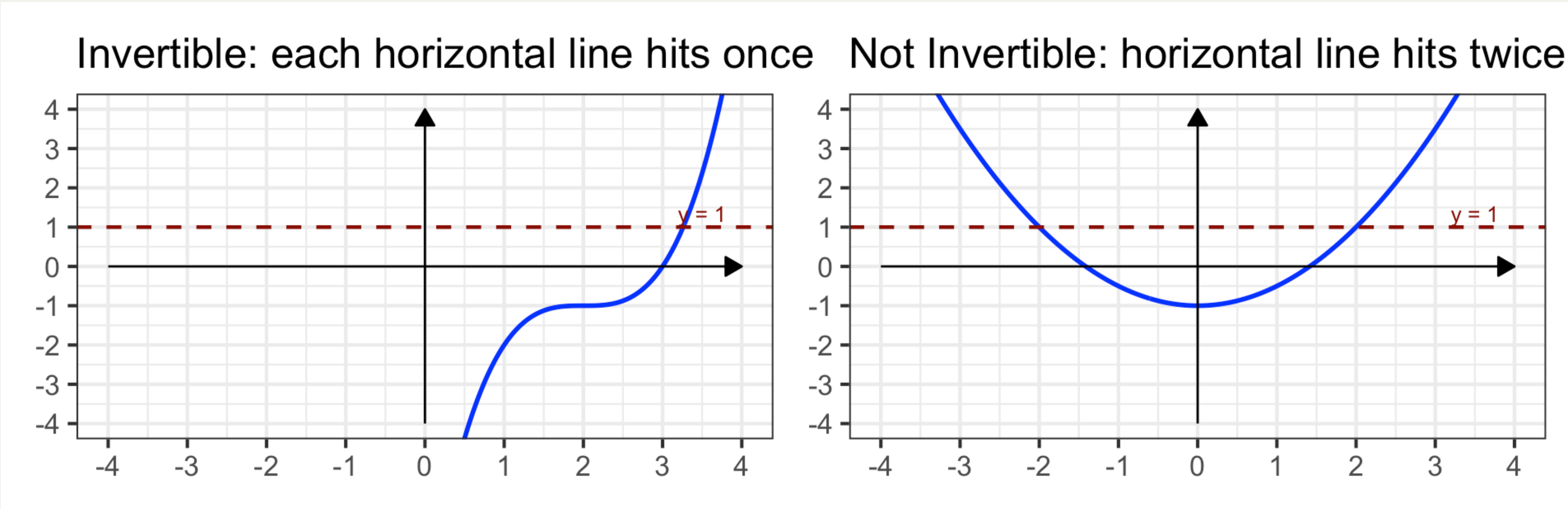
The inverse of a function must itself be a function. That is, the "inverse" must pass the *vertical line test*.

This means only functions whose graphs pass both the *vertical line test* and the *horizontal line test* are invertible.

Definition (Horizontal Line Test): A function is *invertible* if every horizontal line drawn intersects its graph at most once.



Which Functions Have Inverses?

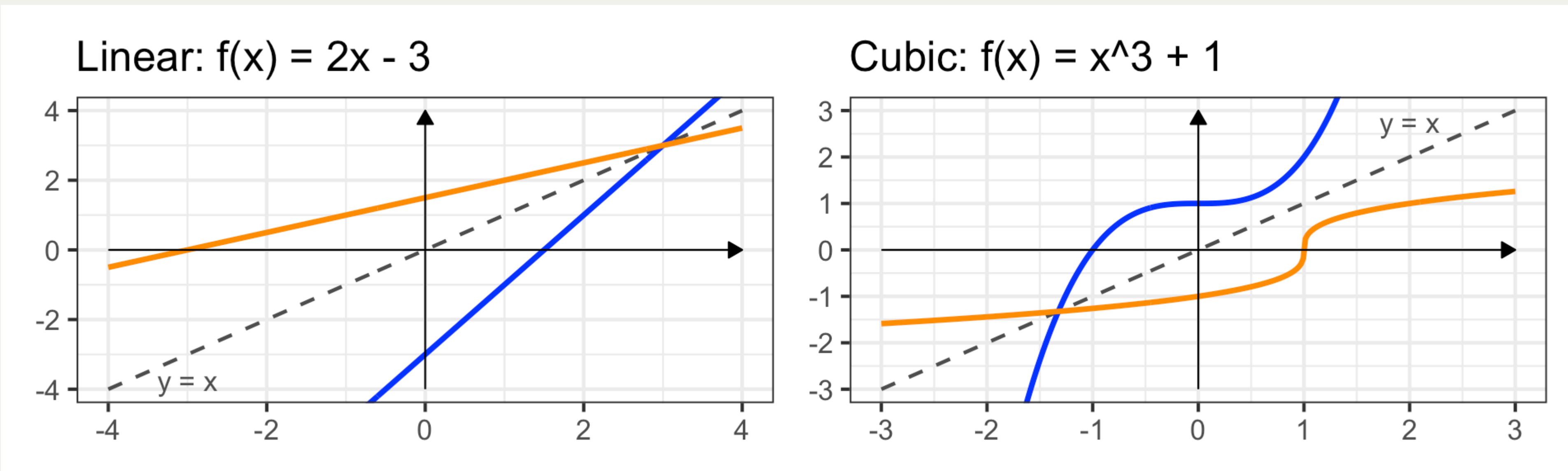


The dashed red line $y = 1$ intersects the left graph exactly once – that output value corresponds to exactly one input. On the right graph it intersects twice – so the "inverse" would need to assign two outputs to one input, which violates the definition of a function.

Graphs of a Function and Its Inverse

There is a beautiful geometric relationship between a function and its inverse: **their graphs are reflections of one another across the line $y = x$.**

We'll leverage this geometric relationship when we introduce our procedure for finding the inverse of a function next. We'll literally *swap* the roles of the x and y variables, which is exactly what reflection across $y = x$ does to a point (a, b) by sending it to (b, a).



Notice that every point (a, b) on f (blue) has a corresponding point (b, a) on f^{-1} (orange) – and both lie symmetrically on either side of the dashed line $y = x$.

The Swap-and-Solve Method

Strategy (Swap-and-Solve): To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
2. Swap every x with y and every y with x.
3. Solve the resulting equation for y, if possible.
4. Replace y with $f^{-1}(x)$.

Example: Use the swap-and-solve method to find the inverse of $f(x) = 3x + 6$.

Solution.

$$f(x) = 3x + 6$$

The Swap-and-Solve Method

Strategy (Swap-and-Solve): To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
2. Swap every x with y and every y with x.
3. Solve the resulting equation for y, if possible.
4. Replace y with $f^{-1}(x)$.

Example: Use the swap-and-solve method to find the inverse of $f(x) = 3x + 6$.

Solution.

$$\begin{aligned} f(x) &= 3x + 6 \\ \implies y &= 3x + 6 \end{aligned}$$

The Swap-and-Solve Method

Strategy (Swap-and-Solve): To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
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Example: Use the swap-and-solve method to find the inverse of $f(x) = 3x + 6$.

Solution.

$$
\begin{aligned}
f(x) &= 3x + 6 \\
\implies y &= 3x + 6 \\
\overset{\text{Swap and Solve}}{\implies} \quad x &= 3y + 6
\end{aligned}
$$

The Swap-and-Solve Method

Strategy (Swap-and-Solve): To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
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Example: Use the swap-and-solve method to find the inverse of $f(x) = 3x + 6$.

Solution.

$$
\begin{aligned}
f(x) &= 3x + 6 \\
\implies y &= 3x + 6 \\
\xrightarrow{\text{Swap and Solve}} {\color{blue}x} &= 3{\color{blue}y} + 6 \\
\implies 3y + 6 &= x
\end{aligned}
$$

The Swap-and-Solve Method

Strategy (Swap-and-Solve): To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
2. Swap every x with y and every y with x.
3. Solve the resulting equation for y, if possible.
4. Replace y with $f^{-1}(x)$.

Example: Use the swap-and-solve method to find the inverse of $f(x) = 3x + 6$.

Solution.

$$\begin{aligned} f(x) &= 3x + 6 \\ \implies y &= 3x + 6 \\ \xRightarrow{\text{Swap and Solve}} \quad {\color{blue}x} &= 3{\color{blue}y} + 6 \\ \implies 3y + 6 &= x \\ \implies 3y &= x - 6 \end{aligned}$$

The Swap-and-Solve Method

Strategy (Swap-and-Solve): To find the inverse of an invertible function $f(x)$:

1. Replace $f(x)$ with y.
2. Swap every x with y and every y with x.
3. Solve the resulting equation for y, if possible.
4. Replace y with $f^{-1}(x)$.

Example: Use the swap-and-solve method to find the inverse of $f(x) = 3x + 6$.

Solution.

$$\begin{aligned} f(x) &= 3x + 6 \\ \implies y &= 3x + 6 \\ \xRightarrow{\text{Swap and Solve}} \quad {\color{blue}x} &= 3{\color{blue}y} + 6 \\ \implies 3y + 6 &= x \\ \implies 3y &= x - 6 \\ \implies y &= \frac{x-6}{3} \end{aligned}$$

So $\boxed{f^{-1}(x) = \dfrac{x-6}{3}}$.

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$f(x) = \frac{2x}{x+4}$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$\begin{aligned} f(x) &= \frac{2x}{x+4} \\ \implies y &= \frac{2x}{x+4} \end{aligned}$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$\begin{aligned} f(x) &= \frac{2x}{x+4} \\ \implies y &= \frac{2x}{x+4} \\ \overset{\text{Swap x and y}}{\implies} \quad {\color{blue}x} &= \frac{2{\color{blue}y}}{{\color{blue}y}+4} \end{aligned}$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$
\begin{aligned}
f(x) &= \frac{2x}{x+4} \\
\implies y &= \frac{2x}{x+4} \\
\overset{\text{Swap x and y}}{\implies} \quad \color{blue}{x} &= \frac{2\color{blue}{y}}{\color{blue}{y}+4} \\
\implies x(y+4) &= 2y
\end{aligned}
$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$\begin{aligned}
f(x) &= \frac{2x}{x+4} \\
\implies y &= \frac{2x}{x+4} \\
\overset{\text{Swap x and y}}{\implies} \quad {\color{blue}x} &= \frac{2{\color{blue}y}}{{\color{blue}y}+4} \\
\implies x(y+4) &= 2y \\
\implies xy + 4x &= 2y
\end{aligned}$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$
\begin{aligned}
f(x) &= \frac{2x}{x+4} \\
\implies y &= \frac{2x}{x+4} \\
\overset{\text{Swap x and y}}{\implies} \color{blue}{x} &= \frac{2\color{blue}{y}}{\color{blue}{y}+4} \\
\implies x(y+4) &= 2y \\
\implies xy + 4x &= 2y \\
\implies xy &= 2y - 4x
\end{aligned}
$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$\begin{aligned} f(x) &= \frac{2x}{x+4} \\ \implies y &= \frac{2x}{x+4} \\ \overset{\text{Swap x and y}}{\implies} \quad x &= \frac{2y}{y+4} \\ \implies x(y+4) &= 2y \\ \implies xy + 4x &= 2y \\ \implies xy &= 2y - 4x \\ \implies xy - 2y &= -4x \end{aligned}$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$\begin{aligned} f(x) &= \frac{2x}{x+4} \\ \implies y &= \frac{2x}{x+4} \\ \overset{\text{Swap x and y}}{\implies} x &= \frac{2y}{y+4} \\ \implies x(y+4) &= 2y \\ \implies xy + 4x &= 2y \\ \implies xy &= 2y - 4x \\ \implies xy - 2y &= -4x \\ \implies y(x-2) &= -4x \end{aligned}$$

Completed Example 4

Example: Use the swap-and-solve method to find the inverse of $f(x) = \dfrac{2x}{x+4}$.

Solution.

$$\begin{aligned} f(x) &= \frac{2x}{x+4} \\ \implies y &= \frac{2x}{x+4} \\ \overset{\text{Swap x and y}}{\implies} \quad x &= \frac{2y}{y+4} \\ \implies x(y+4) &= 2y \\ \implies xy + 4x &= 2y \\ \implies xy &= 2y - 4x \\ \implies xy - 2y &= -4x \\ \implies y(x-2) &= -4x \\ \implies y &= \frac{-4x}{x-2} \end{aligned}$$

So $\boxed{f^{-1}(x) = \dfrac{-4x}{x-2}}$.

Finding Inverses Algebraically Practice

Try It! 3: Use the swap-and-solve method to find $g^{-1}(x)$ if $g(x) = \dfrac{x-3}{9}$.

Try It! 4: Use the swap-and-solve method to find $h^{-1}(x)$ if $h(x) = \dfrac{2x-5}{x+2}$.

Applied Problem

Problem: To convert x degrees Celsius to y degrees Fahrenheit, we use

$$f(x) = \frac{9}{5}x + 32$$

Find $f^{-1}(x)$, if it exists, and explain what it means.

Solution.

$$f(x) = \frac{9}{5}x + 32$$

Applied Problem

Problem: To convert x degrees Celsius to y degrees Fahrenheit, we use

$$f(x) = \frac{9}{5}x + 32$$

Find $f^{-1}(x)$, if it exists, and explain what it means.

Solution.

$$\begin{aligned} f(x) &= \frac{9}{5}x + 32 \\ \implies y &= \frac{9}{5}x + 32 \end{aligned}$$

Applied Problem

Problem: To convert x degrees Celsius to y degrees Fahrenheit, we use

$$f(x) = \frac{9}{5}x + 32$$

Find $f^{-1}(x)$, if it exists, and explain what it means.

Solution.

$$\begin{aligned} f(x) &= \frac{9}{5}x + 32 \\ \Longrightarrow \quad y &= \frac{9}{5}x + 32 \\ \overset{\text{Swap x and y}}{\Longrightarrow} \quad {\color{blue}x} &= \frac{9}{5}{\color{blue}y} + 32 \end{aligned}$$

Applied Problem

Problem: To convert x degrees Celsius to y degrees Fahrenheit, we use

$$f(x) = \frac{9}{5}x + 32$$

Find $f^{-1}(x)$, if it exists, and explain what it means.

Solution.

$$\begin{aligned} f(x) &= \frac{9}{5}x + 32 \\ \implies y &= \frac{9}{5}x + 32 \\ \overset{\text{Swap x and y}}{\implies} \color{blue}{x} &= \frac{9}{5}\color{blue}{y} + 32 \\ \implies \frac{9}{5}y + 32 &= x \end{aligned}$$

Applied Problem

Problem: To convert x degrees Celsius to y degrees Fahrenheit, we use

$$f(x) = \frac{9}{5}x + 32$$

Find $f^{-1}(x)$, if it exists, and explain what it means.

Solution.

$$\begin{aligned} f(x) &= \frac{9}{5}x + 32 \\ \implies y &= \frac{9}{5}x + 32 \\ \overset{\text{Swap x and y}}{\implies} {\color{blue}x} &= \frac{9}{5}{\color{blue}y} + 32 \\ \implies \frac{9}{5}y + 32 &= x \\ \implies \frac{9}{5}y &= x - 32 \end{aligned}$$

Applied Problem

Problem: To convert x degrees Celsius to y degrees Fahrenheit, we use

$$f(x) = \frac{9}{5}x + 32$$

Find $f^{-1}(x)$, if it exists, and explain what it means.

Solution.

$$\begin{aligned} f(x) &= \frac{9}{5}x + 32 \\ \implies y &= \frac{9}{5}x + 32 \\ \overset{\text{Swap x and y}}{\implies} \quad x &= \frac{9}{5}y + 32 \\ \implies \frac{9}{5}y + 32 &= x \\ \implies \frac{9}{5}y &= x - 32 \\ \implies y &= \frac{5}{9}(x - 32) \end{aligned}$$

Interpretation: $f^{-1}(x)$ converts x degrees *Fahrenheit* back to degrees *Celsius*. For example, $f^{-1}(212) = \frac{5}{9}(212 - 32) = \frac{5}{9}(180) = 100$ degrees Celsius – the boiling point of water, as expected.

This also answers our motivating questions from the start of class: since $f(0) = 32$, we have $f^{-1}(32) = 0$ – freezing point in Fahrenheit corresponds to $0^\circ C$. And solving $f(c) = 40$ gives the Celsius temperature that corresponds to $40^\circ F$.

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `7. Inverses of Functions`

Task: Use the swap-and-solve method to find the inverse of $f(x) = 4x - 8$. Then verify your answer by checking that $f\left(f^{-1}(x)\right) = x$.

Summary and Next Time...

Ideas From Today

- f and g are **inverses** if $f(g(x)) = x$ and $g(f(x)) = x$.
- Check invertibility using the **horizontal line test** – each output value must correspond to exactly one input.
- Use **swap-and-solve** to find f^{-1}: replace $f(x)$ with y, swap x and y, solve for y.
- $f^{-1}(x)$ means the *inverse* of f, not $\frac{1}{f(x)}$.
- Inverse functions arise naturally whenever we need to *reverse* a process – like converting units back to their original form.

Looking Ahead

- Inverse functions will appear throughout the rest of the course. For every function class we study, we will ask whether and how to invert functions of that class.
- You have all the tools you need to find inverses of *linear*, *polynomial*, and *rational* functions at this time. Some classes of function will require special inverses though.
  - The inverse of an *exponential* function is a *logarithm* and the inverse of a *trigonometric* function requires special *inverse-trigonometric* functions, for example.

Next Time:

Linear Functions and Linear Equations

Homework:

Complete Homework 5 on MyOpenMath