MAT 142: Book Functions, Transformations, and Landmarks

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Reminders

Welcome back from Exam I!

Over those first few weeks of our course, we focused on truly foundational items.

- Shoring up our algebra background
- Interval notation
- Domain and range
- Composition of functions

We also encountered the notion of the *limit*, which will be a foundational tool in Calculus as well as one we work to become comfortable with here in PreCalculus.

We'll continue with a few more foundational topics before moving into exploring specific function classes for the remainder of our semester.

Try the following warm-up problems to get your algebra legs back.

Problem 1: Consider $f\left(x\right) = x^2$. Find each of the following.

- $f\left(3\right)$
- $f\left(4\right) - 6$
- $2 \cdot f\left(3\right) + 1$
- $f\left(2x\right)$
- $f\left(x-h\right)$
- $f\left(x-h\right) + k$

Motivating Application

A chemist is running an experiment with a theoretical model $f(t)$ measuring the expected temperature (°C) of a liquid at time t (minutes) into a reaction.

They start a **second** reaction 5 minutes later. In this second reaction, the temperature at any given time into the reaction is expected to be **triple** the temperature in the original reaction at that same elapsed time.

Construct a function related to $f(t)$ which models the temperature of the second reaction at t minutes after the start of the *initial* reaction.

Before we work through this, consider the following:

- What is the practical domain of $f(t)$?
- What is the practical domain of the function you are constructing?
- How do you expect the *shape* of the new function to relate to the shape of $f(t)$?

We'll return to this application at the end of today's class.

Objectives

Today has two connected halves.

First half – Book Functions and Transformations:

- Store mental images of the graphs of several foundational function classes.
- Develop the notions of vertical and horizontal shifts, stretches, compressions, and reflections.
- Combine several transformations to construct graphs of more general functions.

Second half – Landmarks on Graphs:

- Identify roots (x-intercepts) and the y-intercept from a graph.
- Identify local minima and maxima.
- Identify intervals of increase, decrease, and constancy.

While this second half consists of things we've seen/done earlier this semester, we'll bring it all back as a refresher.

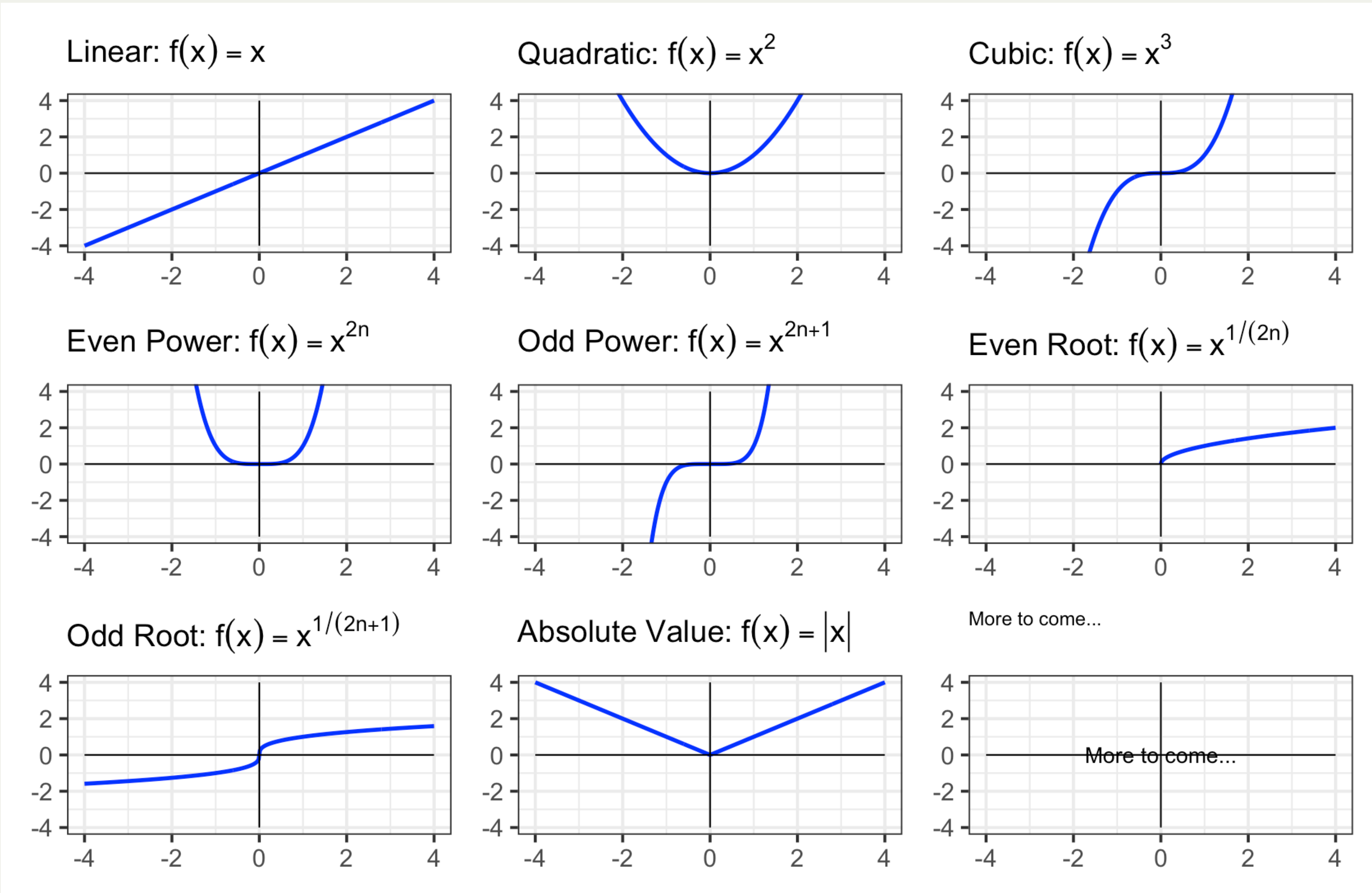
Book Functions

As we explore functions in greater depth, it is helpful to have the general *shape* of a function's graph committed to memory.

We'll refer to these foundational functions as **book functions** – they are the basic building blocks from which more general functions are constructed.

The graphs of our current book functions appear on the next slide. **You should commit these shapes to memory.** We will add to this list later in the course.

Book Functions



Book Functions – Patterns to Notice

- Any **linear** function has a graph that is a straight line.
- Any **power function with even exponent** (x^{2n}) has a "U"-shaped graph, symmetric about the y-axis.
- Any **power function with odd exponent** (x^{2n+1}) has an elongated shape similar to $f(x) = x^3$, symmetric about the origin.
- Any **even root function** ($x^{1/(2n)}$) has domain $[0, \infty)$ and a "half-bowl" shape.
- Any **odd root function** ($x^{1/(2n+1)}$) has an elongated "S"-shape defined on all of $\mathbb{R}$.
- The **absolute value function** has a "V"-shape with vertex at the origin.

This knowledge gives us great leverage in understanding the shapes of more general functions via **transformations**.

Transformations of Functions

We can modify a book function in four fundamental ways:

- **Vertical shift:** move the graph up or down
- **Horizontal shift:** move the graph left or right
- **Vertical stretch/compression/reflection:** make the graph taller/shorter, or flip it over the x-axis
- **Horizontal stretch/compression/reflection:** speed up or slow down the function, or flip it over the y-axis

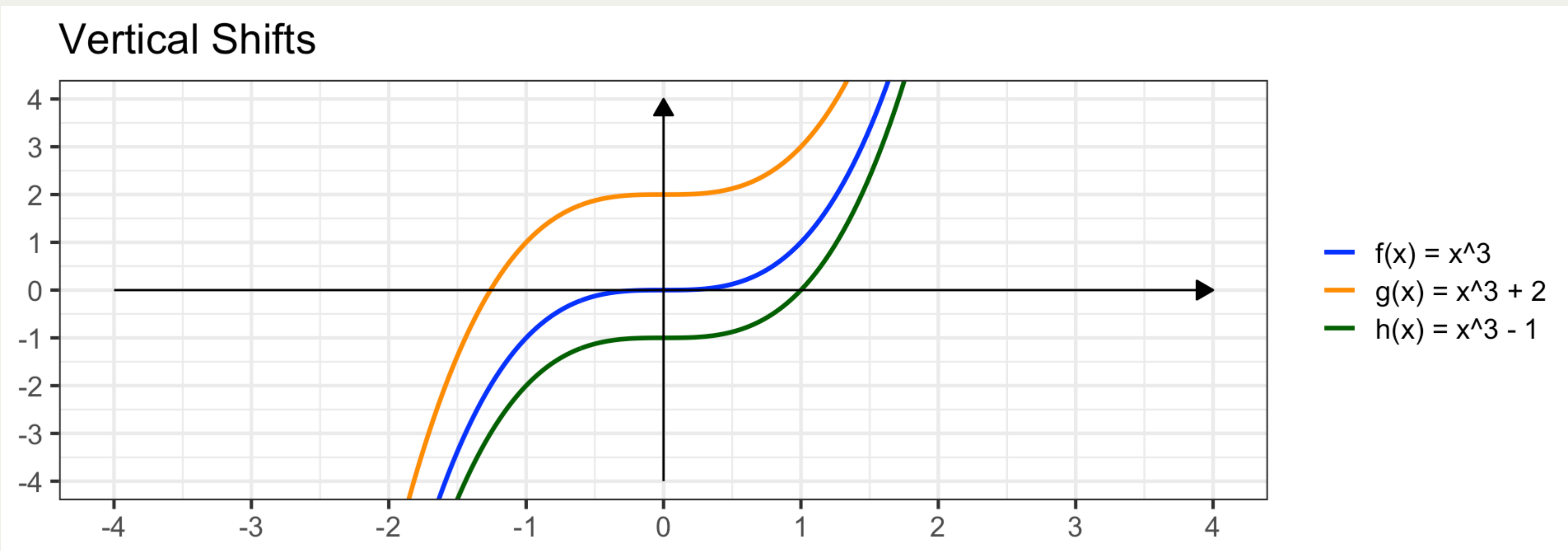
We'll use $f(x) = x^3$ as our running example throughout.

Vertical Shifts

Definition (Vertical Shift): If $g(x) = f(x) + k$, then the graph of g is the graph of f shifted **up by k units** (or down if $k < 0$).

Completed Example 1: Consider $f(x) = x^3$, $g(x) = x^3 + 2$, and $h(x) = x^3 - 1$.

An interactive version of this plot is available on Desmos.



Horizontal Shifts

Definition (Horizontal Shift): If $g(x) = f(x - h)$, then the graph of g is the graph of f shifted **right by h units** (or left if $h < 0$).

Horizontal Shifts are Counter-Intuitive

This is counterintuitive – subtracting inside the argument shifts *right*, adding shifts *left*.

Completed Example 2: Consider $f(x) = x^3$, $g(x) = (x - 2)^3$, and $h(x) = (x + 1)^3$.

An interactive version is available on Desmos.

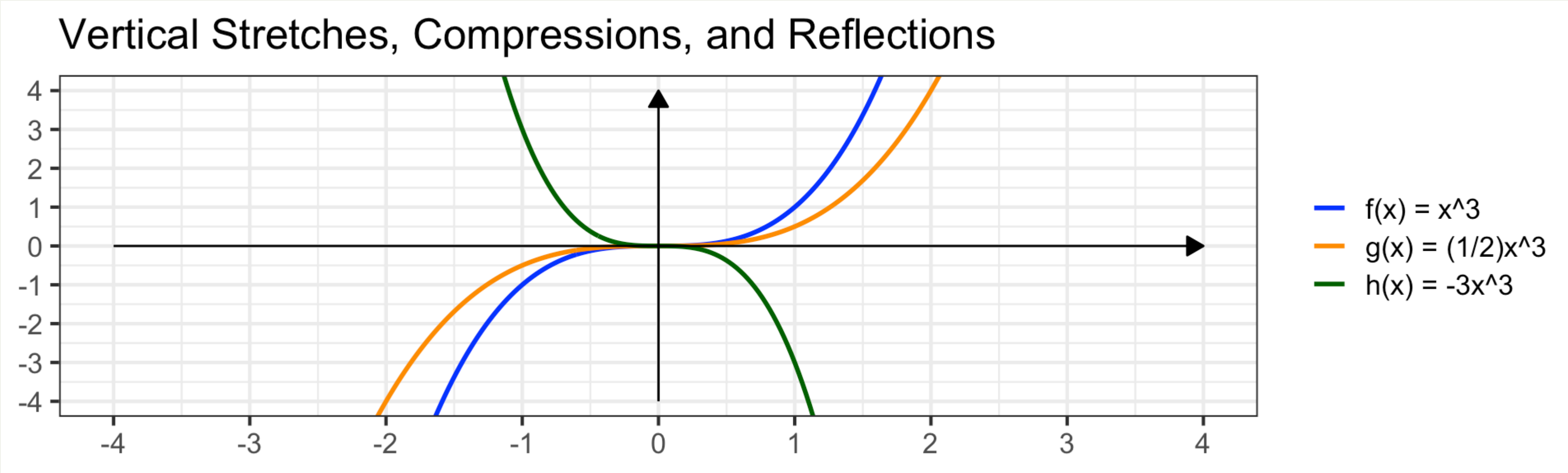
Vertical Stretches, Compressions, and Reflections

Definition (Vertical Stretch): If $g(x) = a \cdot f(x)$, then:

- If $a > 1$: graph is stretched **taller and narrower** by factor a.
- If $0 < a < 1$: graph is compressed **shorter and wider** by factor a.
- If $a < 0$: graph is **reflected over the x-axis**, then stretched/compressed by $|a|$.

Completed Example 3: Consider $f(x) = x^3$, $g(x) = \frac{1}{2}x^3$, and $h(x) = -3x^3$.

An interactive version is available on Desmos.


Vertical Stretches, Compressions, and Reflections
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4
f(x) = x^3
g(x) = (1/2)x^3
h(x) = -3x^3

Horizontal Stretches, Compressions, and Reflections

Definition (Horizontal Stretch): If $g(x) = f(b \cdot x)$, then:

- If $b > 1$: graph is **compressed horizontally** (the function "speeds up").
- If $0 < b < 1$: graph is **stretched horizontally** (the function "slows down").
- If $b < 0$: graph is **reflected over the y-axis**, then stretched/compressed by $|b|$.

Completed Example 4: We use $f(x) = \sin(x)$ as the base function here since the horizontal effects are easier to see. Compare $g(x) = \sin(2x)$ and $h(x) = \sin\left(-\frac{1}{2}x\right)$.

An interactive version is available on Desmos.


Horizontal Stretches, Compressions, and Reflections
2
1
0
-1
-2
-9
-6
-3
0
3
6
9
f(x) = sin(x)
g(x) = sin(2x)
h(x) = sin(-x/2)

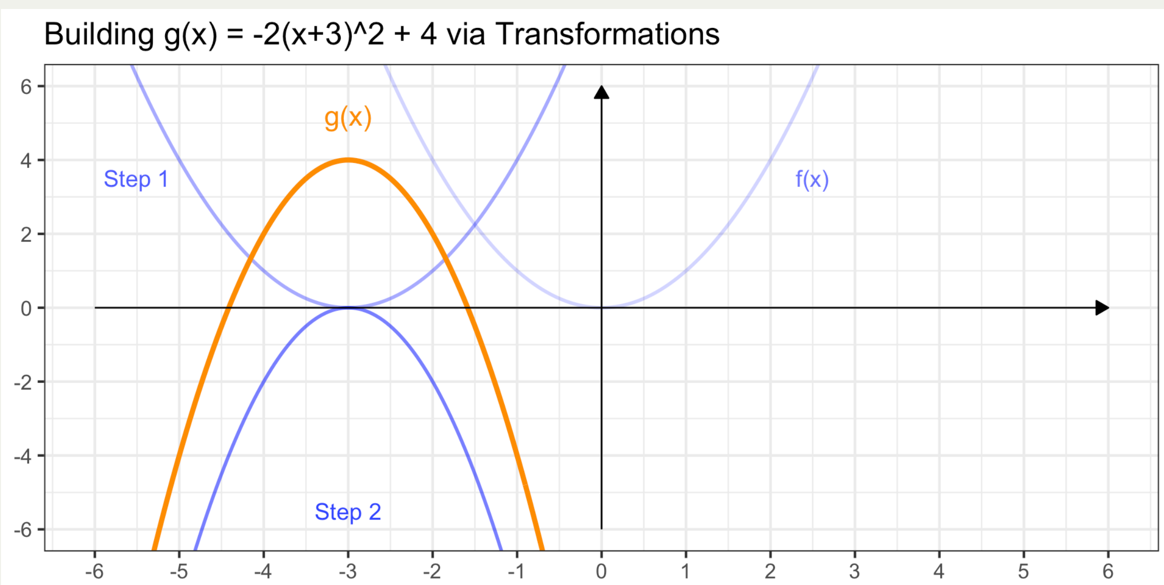
Putting It All Together

We aren't restricted to a single transformation at a time. We can **combine** transformations. The order in which they are applied follows the order of operations.

Completed Example 5: Use transformations to plot $g(x) = -2(x+3)^2 + 4$.

Solution. Compare to the book function $f(x) = x^2$. The transformations occur in this order:

1. Shift **left** by 3 units: $f(x+3) = (x+3)^2$
2. Reflect over x-axis and stretch vertically by 2: $-2(x+3)^2$
3. Shift **up** by 4 units: $g(x) = -2(x+3)^2 + 4$



The faded blue curves show the intermediate stages. The final function $g(x)$ is in orange. Notice the vertex is now at $(-3, 4)$.

Important

Order of operations determines the way in which the transformations are applied. That order matters.

Transformations Practice

Try It! 1: Use your knowledge of transformations to sketch the graph of $g\left(x\right) = 5\sqrt{\left(x+4\right)} + 2$.

Identify the book function, describe each transformation in order, and sketch the result.

Try It! 2: Construct a function $g\left(x\right)$ whose graph looks like $f\left(x\right) = x^{1/3}$, but:

- Shifted **right** by 2 units
- Compressed vertically by a factor of $\frac{1}{4}$
- Shifted **down** by 5 units

Provide both the algebraic definition of $g\left(x\right)$ and sketch an approximate graph.

Landmarks on Graphs of Functions

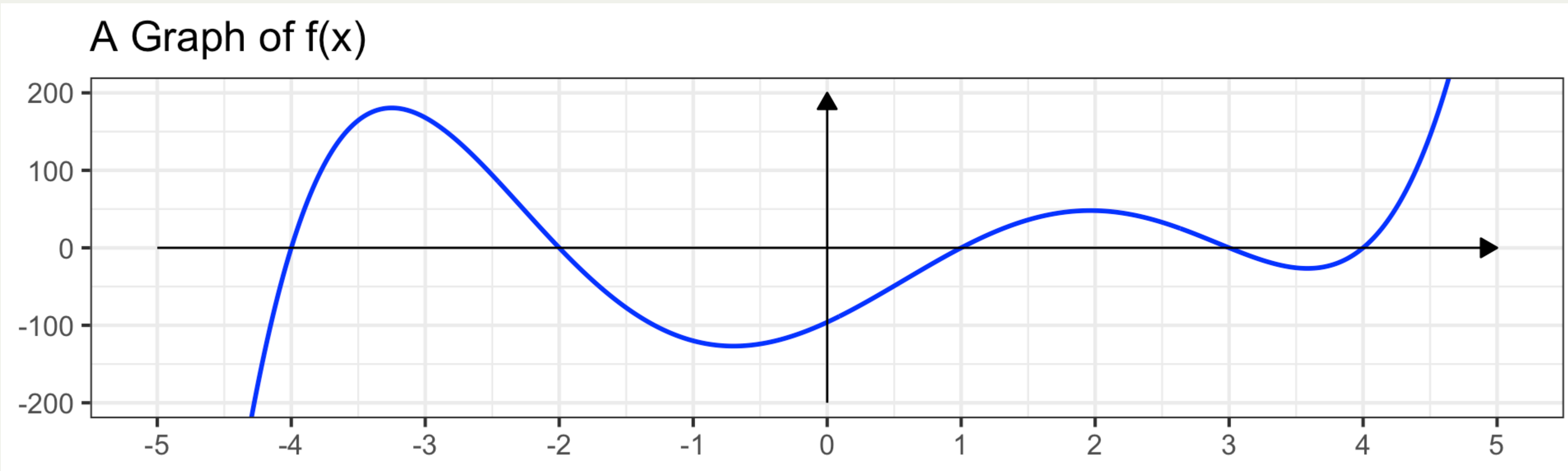
We've seen function landmarks before (Day 3), but we'll revisit them here since they now have a richer context with transformations in mind.

For any function, we want to be able to identify:

- **Roots** (x-intercepts): inputs where $f(x) = 0$
- **y-intercept**: the point $(0, f(0))$
- **Local maxima**: peaks where the function transitions from increasing to decreasing
- **Local minima**: valleys where the function transitions from decreasing to increasing
- **Intervals of increase/decrease**: where the graph is rising or falling (left to right)

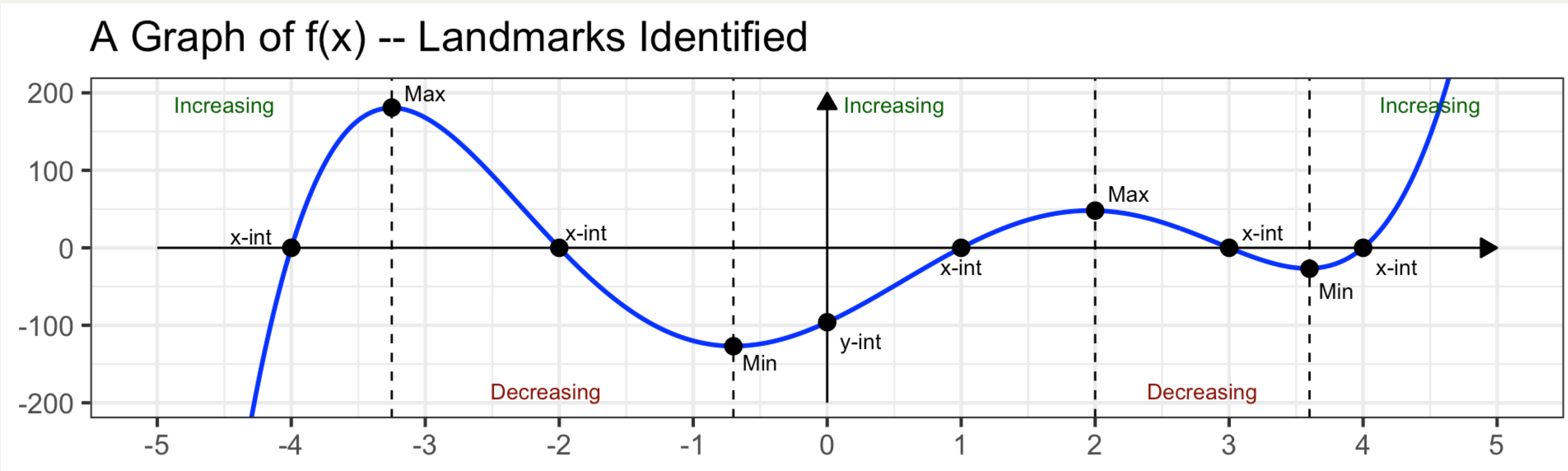
Completed Example 6

Problem: Find the roots, y-intercept, local maxima and minima, and intervals of increase and decrease for $f(x)$
.



Completed Example 6

Problem: Find the roots, y-intercept, local maxima and minima, and intervals of increase and decrease for $f(x)$
.



Solution. By inspecting the annotated graph (approximations, not exact values):

Motivating Application – Revisited

Problem: A chemist has model $f(t)$ for a reaction's temperature. A second reaction starts **5 minutes later**, and its temperature is expected to be **triple** the first reaction's temperature at each time point into the reaction.

Construct a function modeling the second reaction's temperature at time t minutes after the *start of the first* reaction.

Solution. We need two transformations applied to f:

1. The second reaction starts 5 minutes later, so we shift **right by 5**: the time into the second reaction at clock-time t is $t-5$, giving $f(t-5)$.
2. The temperature is **triple** at each time point, so we stretch vertically by $\mathbf{3}$, resulting in $3f(t-5)$.

$$\boxed{g(t) = 3f(t-5)}$$

The practical domain is $t \geq 5$, and the shape of g is the same as f, just shifted right and stretched vertically.

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `6. Graphing: Book Functions and Transformations`

Task: Describe the transformations applied to the book function $f(x) = \sqrt{x}$ to obtain $g(x) = -3\sqrt{(x-2)} + 1$. List them in order and describe the effect of each on the graph.

Summary and Next Time…

Ideas From Today

- **Book functions** are foundational shapes to memorize: linear, quadratic, cubic, even/odd powers, even/odd roots, and absolute value.
- $f(x) + k$: vertical shift up by k
- $f(x - h)$: horizontal shift right by h
- $a \cdot f(x)$: vertical stretch ($a > 1$), compression ($0 < a < 1$), or reflection ($a < 0$)
- $f(b \cdot x)$: horizontal compression ($b > 1$), stretch ($0 < b < 1$), or reflection ($b < 0$)
- Transformations combine following the order of operations.
- Landmarks: roots, y-intercept, local maxima/minima, intervals of increase/decrease.

Resources

- Vertical shifts (Desmos)
- Horizontal shifts (Desmos)
- Vertical stretches (Desmos)
- Horizontal stretches (Desmos)

Next Time:
Inverse Functions
Homework:
Start Homework 5 on MyOpenMath