MAT 142: Trig Functions, Inverses, and Equations

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June 12, 2026

Reminders and Warm-Up

Last class we introduced angles, radian measure, coterminal angles, and used the unit circle to evaluate $\sin()$, $\cos()$, and $\tan()$. Try the following warm-up problems.

Problem 1: Evaluate the following using the unit circle.

$$(a)\ \ \sin\left(\frac{7\pi}{6}\right) \qquad (b)\ \ \tan(120^\circ) \qquad (c)\ \ \cos(330^\circ)$$

Problem 2: Evaluate the following. Use coterminal angles as needed.

$$(a)\ \ \tan(-135^\circ) \qquad (b)\ \ \cos\left(\frac{27\pi}{6}\right) \qquad (c)\ \ \cos\left(\frac{-15\pi}{3}\right)$$

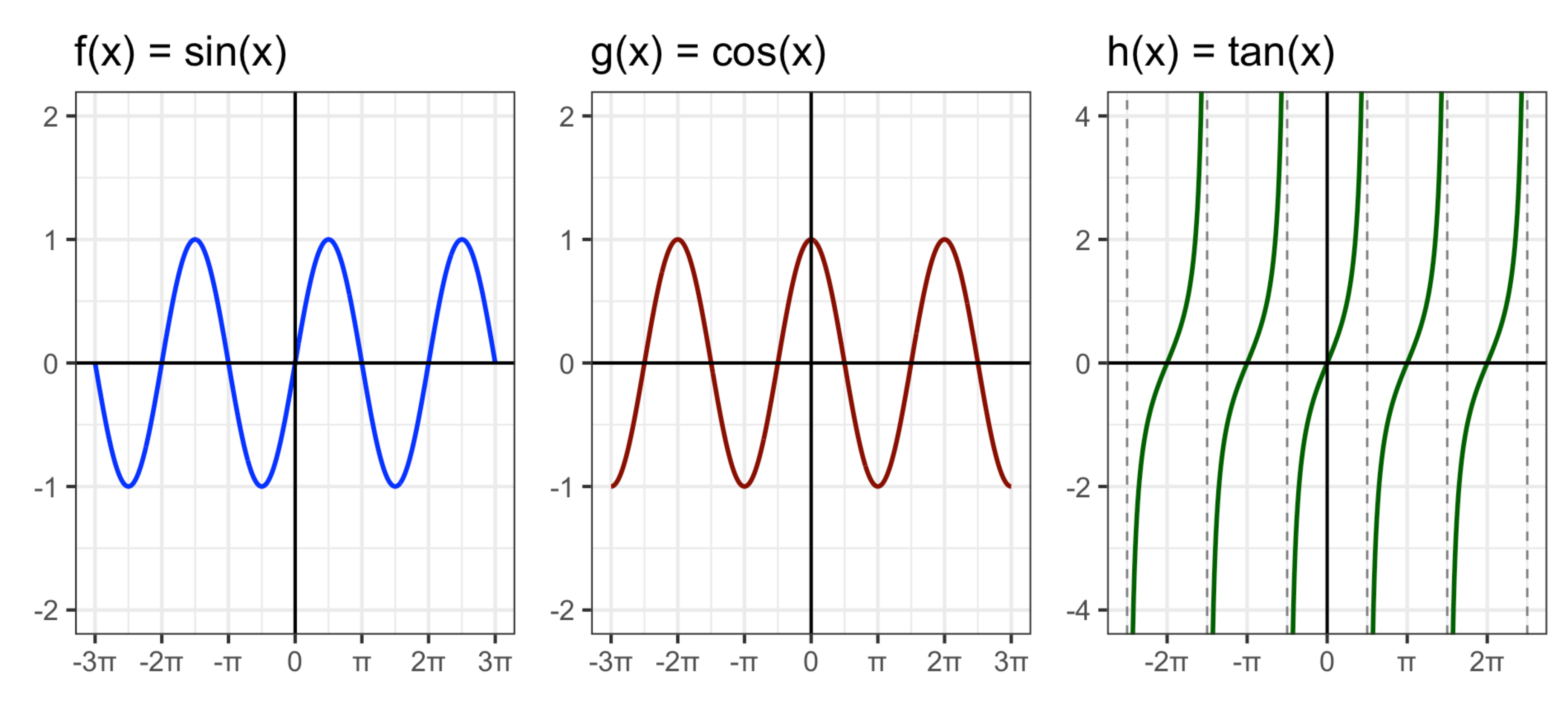
Objectives

Today we study the graphs of the three primary trigonometric functions, their inverses, and how to solve trigonometric equations.

After today's class meeting, you should be able to:

- Identify key features (period, amplitude, phase shift, vertical shift) of transformed sine and cosine functions.
- Evaluate inverse trigonometric functions **arcsin**, **arccos**, and **arctan**.
- Solve trigonometric equations for all solutions using the unit circle.

The Book Trig Functions



Geography of Sine and Cosine

Feature	$\sin(x)$	$\cos(x)$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[-1, 1]$	$[-1, 1]$
Period	2π	2π
y-intercept	$(0, 0)$	$(0, 1)$
Asymptotes	None	None

Tangent behaves differently: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, so it is undefined whenever $\cos(x) = 0$.

- Domain: all $x \neq \frac{\pi}{2} + \pi k$ for any integer k.
- Period: π (half that of sine and cosine).
- Vertical asymptotes at $x = \frac{\pi}{2} + \pi k$.

Transformations of Trig Functions

The general transformed form is $f(x) = A \cdot \text{trig}(b(x-h)) + k$, where:

Parameter	Effect
$\|A\|$	**Amplitude** — vertical stretch factor; range becomes $[k - \|A\|,\ k + \|A\|]$
$A < 0$	Reflection over the midline
h	**Phase shift** — horizontal shift ($h > 0$ shifts right)
k	**Vertical shift** — midline moves to $y = k$
b	**Period** — for $\sin/\cos$: period $= \dfrac{2\pi}{\|b\|}$; for $\tan$: period $= \dfrac{\pi}{\|b\|}$

Always Factor First

If the argument isn't already in the form $b(x-h)$, factor out b before reading off the phase shift. For example, $\sin\left(\frac{1}{2}x - 3\right) = \sin\left(\frac{1}{2}(x-6)\right)$, so the phase shift is 6, not 3.

Transformations and Graphing

Example 1: For $f(x) = -3\sin\left(\frac{1}{2}x - 3\right) + 2$, identify the amplitude, period, phase shift, and vertical shift. Sketch the graph.

Solution. Factor out $b = \frac{1}{2}$ from the argument first.

$$f(x) = -3\sin\left(\frac{1}{2}x - 3\right) + 2$$

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$$\begin{aligned} f(x) &= -3\sin\left(\frac{1}{2}x - 3\right) + 2 \\ &= -3\sin\left(\frac{1}{2}(x - 6)\right) + 2 \end{aligned}$$

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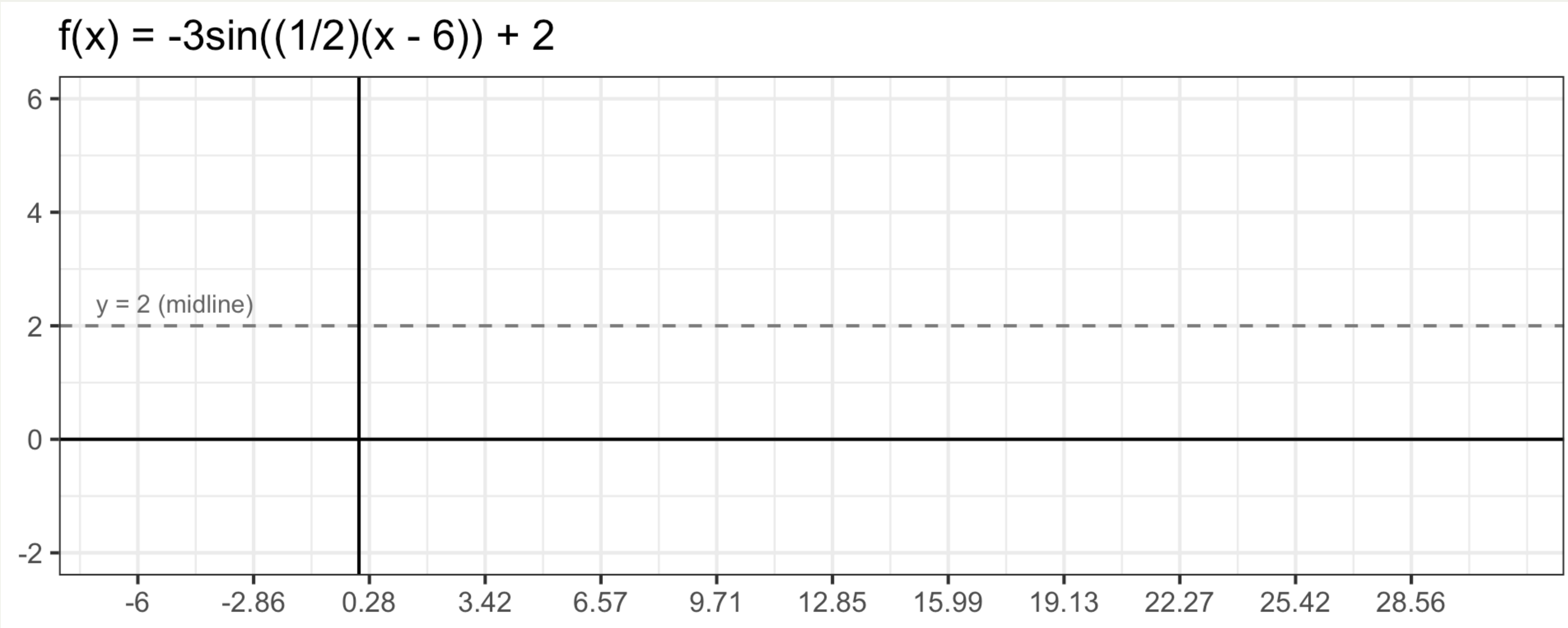
Now we can read off the parameters directly.

- **Amplitude:** $|-3| = 3$ (the wave is also reflected over the midline, since $A < 0$)
- **Phase shift:** 6 units to the right
- **Period:** $\dfrac{2\pi}{1/2} = 4\pi$
- **Vertical shift:** $+2$ (midline at $y = 2$, range $[-1, 5]$)

Transformations and Graphing



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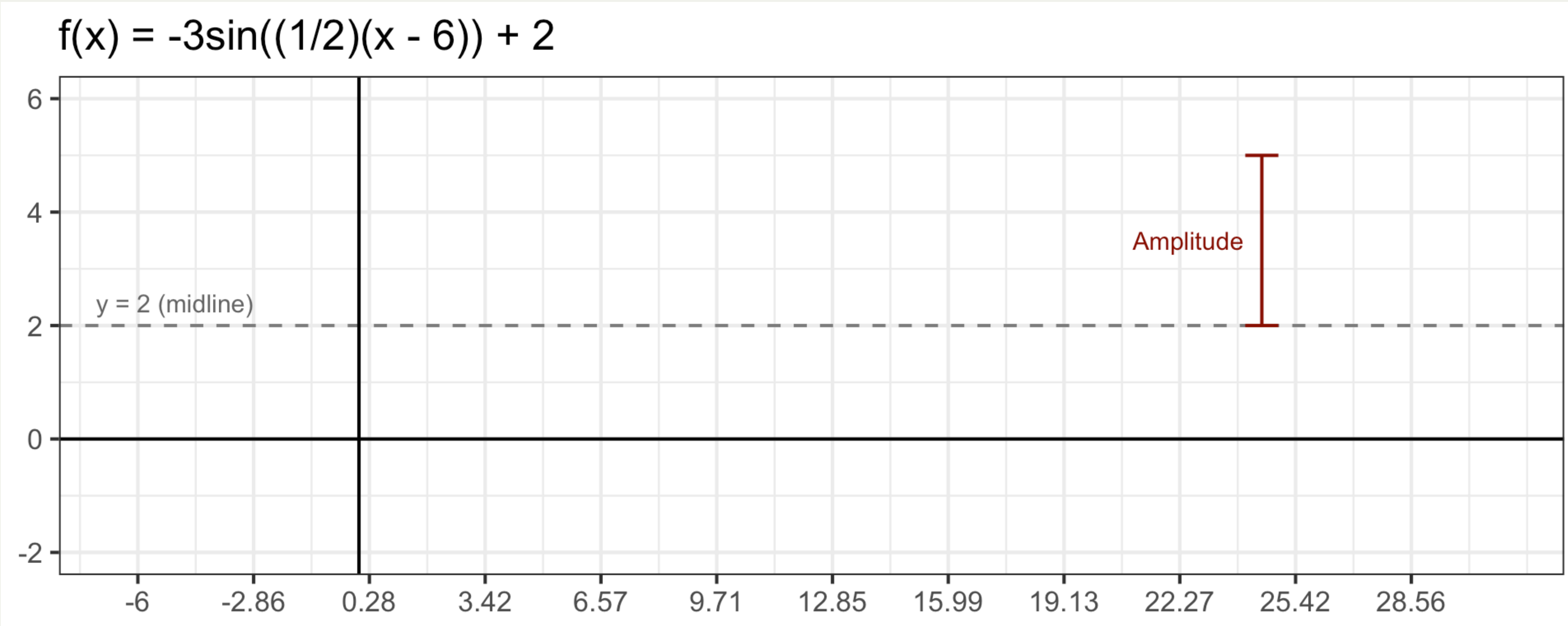

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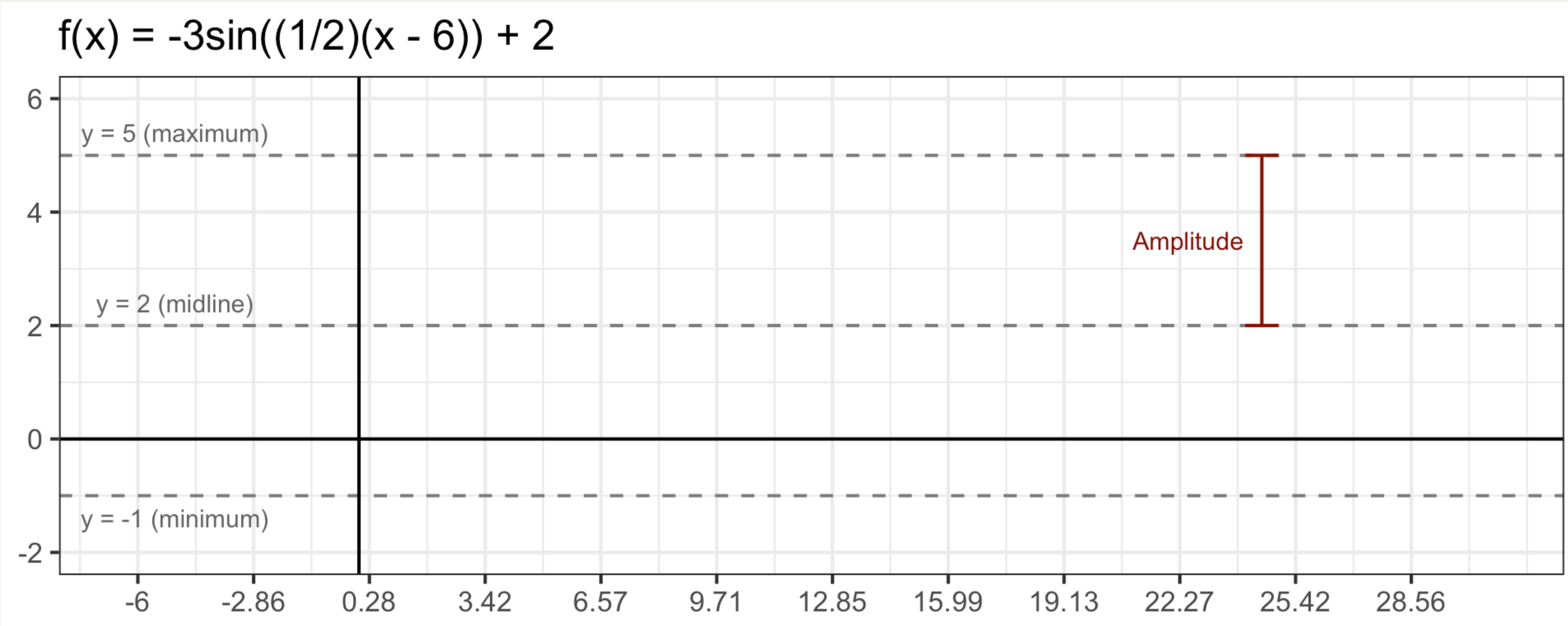

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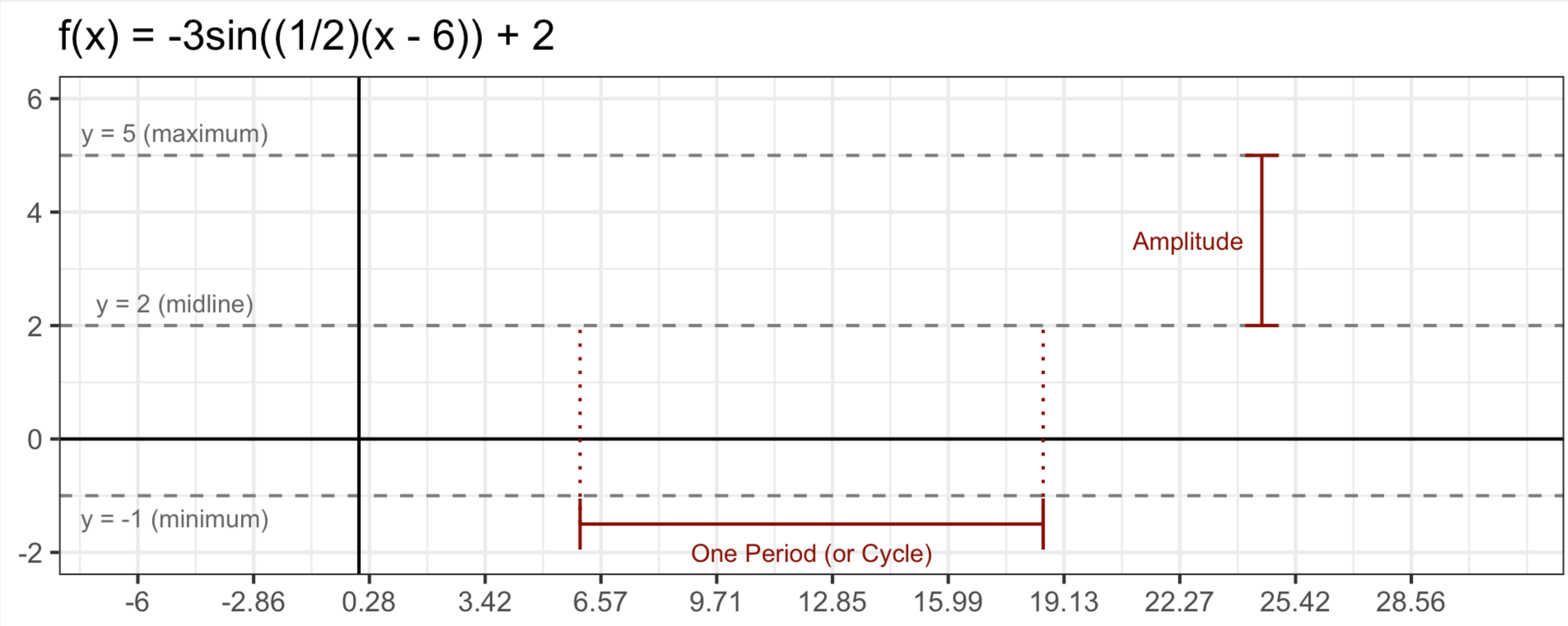

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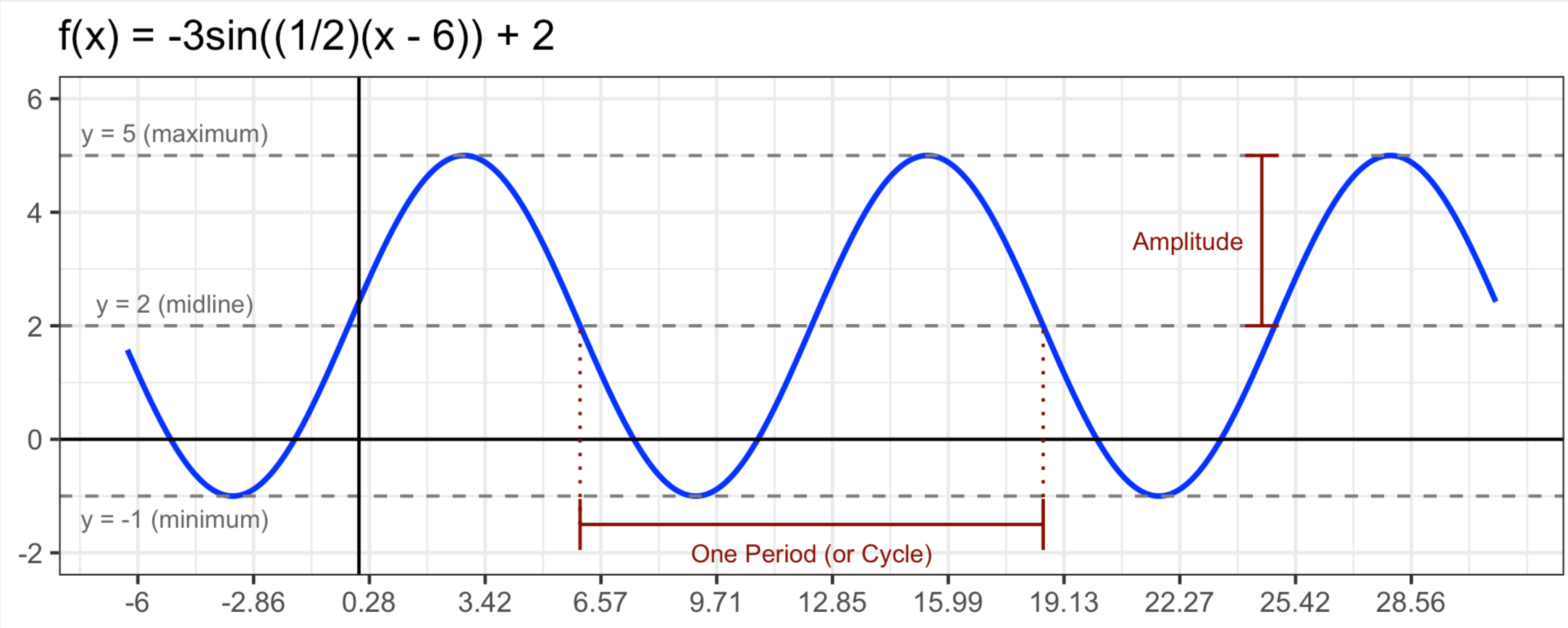

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Wave Function Practice

Try It! 1: For $f(x) = 5\cos(2x+4) - \frac{1}{2}$, identify the amplitude, period, phase shift, and vertical shift.

Remember: Factor out b from the argument before reading off the phase shift.

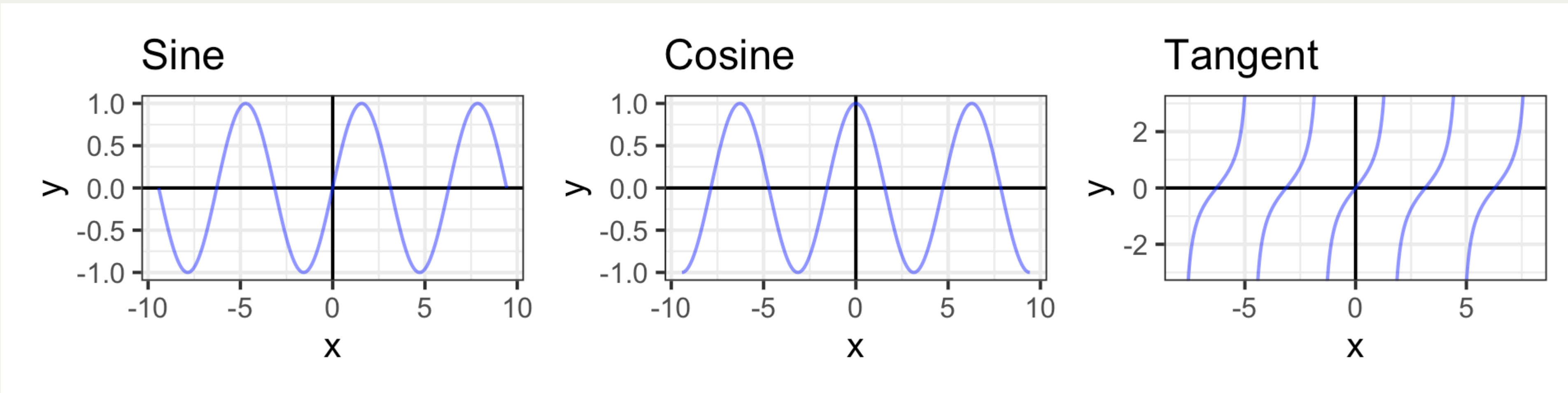
Try It! 2: A wave function of the form $f(x) = A\sin(b(x-h)) + k$ has:

- Range $[-2, 8]$
- Period π
- Passes through the point $(0, -2)$

Sketch the graph and write down the algebraic definition of f.

Inverse Trigonometric Functions

Since $\sin()$, $\cos()$, and $\tan()$ fail the horizontal line test, they are not invertible on their full domains. We restrict each to a domain where it is one-to-one, then define the inverse.



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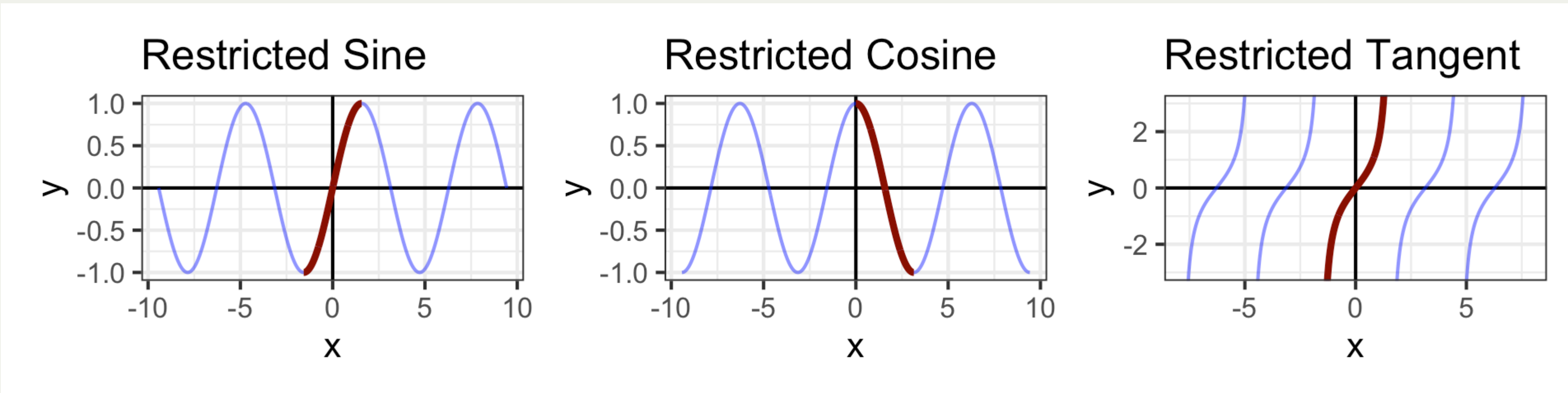

Function	Restricted domain	Inverse	Permissible Inputs	Output range
$\sin(x)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\arcsin(x)$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos(x)$	$[0, \pi]$	$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\tan(x)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\arctan(x)$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Reading inverse trig: $\arcsin(x)$ asks *"what angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has sine equal to x?"* The other two functions work the same way within their respective output ranges.

Both notations $\arcsin(x)$ and $\sin^{-1}(x)$ mean the same thing.

Evaluating Inverse Trig Functions

Example 2: Evaluate each of the following.

$$(a)\ \ \arccos\left(\frac{-1}{2}\right) \qquad (b)\ \ \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad (c)\ \ \arctan(-1)$$

(a) $\arccos\left(\frac{-1}{2}\right)$

Ask: what angle $\theta \in [0, \pi]$ satisfies $\cos(\theta) = \dfrac{-1}{2}$?

From the unit circle, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\frac{2\pi}{3} \in [0, \pi]$.

$$\arccos\left(\frac{-1}{2}\right) = \boxed{\frac{2\pi}{3}}$$

(b) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Ask: what angle $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfies $\sin(\theta) = \dfrac{\sqrt{3}}{2}$?

From the unit circle, $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{3}}$$

(c) $\arctan(-1)$

Ask: what angle $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies $\tan(\theta) = -1$?

From the unit circle, $\tan\left(\frac{-\pi}{4}\right) = -1$ and $\frac{-\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\arctan(-1) = \boxed{\frac{-\pi}{4}}$$

Inverse Trig Practice

Try It! 3: Evaluate each of the following inverse trigonometric functions.

$$(a)\ \ \arccos\left(\frac{\sqrt{2}}{2}\right) \qquad (b)\ \ \arcsin(-1) \qquad (c)\ \ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$(d)\ \ \cos^{-1}(0) \qquad (e)\ \ \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) \qquad (f)\ \ \tan^{-1}(-\sqrt{3})$$

Solving Trigonometric Equations

Trigonometric equations typically have **infinitely many solutions** because **sin**, **cos**, and **tan** are periodic. The general strategy is:

1. Isolate the trigonometric function.
2. Use **arcsin**, **arccos**, or **arctan** to find the reference angle — this gives the principal solution.
3. Use the unit circle to find **all** angles in $[0, 2\pi)$ where the equation holds.
4. Write the **general solution** by adding integer multiples of the period.

Always Give All Solutions

Unless the problem restricts the domain, trigonometric equations have infinitely many solutions. Always append $+2\pi k$ (or $+\pi k$ for tangent) where k is an integer immediately after using the inverse trigonometric function to "*unlock*" the argument.

Solving Trigonometric Equations

Example 3: Solve $2\sin(2x) + \sqrt{3} = 0$.

Solution. Isolate the trigonometric function first.

$$2\sin(2x) + \sqrt{3} = 0$$

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$$\begin{aligned} &2\sin(2x) + \sqrt{3} = 0 \\ \implies &2\sin(2x) = -\sqrt{3} \end{aligned}$$

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Solution. Isolate the trigonometric function first.

$$\begin{aligned} 2\sin(2x) + \sqrt{3} &= 0 \\ \implies 2\sin(2x) &= -\sqrt{3} \\ \implies \sin(2x) &= \frac{-\sqrt{3}}{2} \end{aligned}$$

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Example 3: Solve $2\sin(2x) + \sqrt{3} = 0$.

Solution. Isolate the trigonometric function first. Then apply the appropriate inverse trigonometric function to "*unlock*" the variable expression.

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$$
\begin{aligned}
2\sin(2x) + \sqrt{3} &= 0 \\
\implies 2\sin(2x) &= -\sqrt{3} \\
\implies \sin(2x) &= \frac{-\sqrt{3}}{2} \\
\implies \arcsin(\sin(2x)) &= \arcsin\left(\frac{-\sqrt{3}}{2}\right)
\end{aligned}
$$

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\implies 2x &= \arcsin\left(\frac{-\sqrt{3}}{2}\right) + 2\pi k
\end{aligned}$$

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$$\begin{aligned} 2\sin(2x)+\sqrt{3}&=0\\ \implies 2\sin(2x)&=-\sqrt{3}\\ \implies \sin(2x)&=\frac{-\sqrt{3}}{2}\\ \implies \arcsin(\sin(2x))&=\arcsin\left(\frac{-\sqrt{3}}{2}\right)\\ \implies 2x&=\arcsin\left(\frac{-\sqrt{3}}{2}\right)+2\pi k \end{aligned}$$

Now evaluate $\arcsin\left(\frac{-\sqrt{3}}{2}\right)$ by finding all angles θ where $\sin(\theta)=\frac{-\sqrt{3}}{2}$.

There are two possible solution angles – one in QIII and the other in QIV.

The angle in QIV is $\frac{-\pi}{3}$ (since $\sin\left(\frac{-\pi}{3}\right)=\frac{\sqrt{3}}{2}$).

The angle in QIII is $\frac{-2\pi}{3}$

Solving Trigonometric Equations

Example 3: Solve $2\sin(2x) + \sqrt{3} = 0$.

Solution. Isolate the trigonometric function first. Then apply the appropriate inverse trigonometric function to "*unlock*" the variable expression.

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2\sin(2x) + \sqrt{3} &= 0 \\
\implies 2\sin(2x) &= -\sqrt{3} \\
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\implies \arcsin(\sin(2x)) &= \arcsin\left(\frac{-\sqrt{3}}{2}\right) \\
\implies 2x &= \arcsin\left(\frac{-\sqrt{3}}{2}\right) + 2\pi k \\
\implies 2x = \frac{-2\pi}{3} + 2\pi k \text{ or } 2x &= \frac{-\pi}{3} + 2\pi k
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\implies 2x &= \arcsin\left(\frac{-\sqrt{3}}{2}\right) + 2\pi k \\
\implies 2x = \frac{-2\pi}{3} + 2\pi k \text{ or } 2x &= \frac{-\pi}{3} + 2\pi k \\
\implies \boxed{x = \frac{-\pi}{3} + \pi k} \text{ or } &\boxed{x = \frac{-\pi}{6} + \pi k} \text{ for any integer } k
\end{aligned}$$

Trigonometric Equation Practice

Try It! 4: Solve each of the following trigonometric equations. Give all solutions.

(a) $2\cos(5x-4)-\sqrt{3}=0$

(b) $\tan(x-3)+\sqrt{3}=0$

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `20. Trigonometric Functions and Equations`

Task: Consider $f(x) = 2\sin(3x+\pi) - 1$.

(a) Identify the amplitude, period, phase shift, and vertical shift.

(b) Solve $f(x) = 0$ for all x. *(Hint: isolate* $\sin(\cdots)$*, then find all solutions.)*

Summary and Next Time...

Ideas From Today

- $\sin()$ and $\cos()$ have period 2π, amplitude 1, domain all reals, range $[-1, 1]$. $\tan()$ has period π and vertical asymptotes at $x = \frac{\pi}{2} + \pi k$.
- For $A \cdot \text{trig}(b\,(x - h)) + k$: amplitude $= |A|$, period $= \frac{2\pi}{b}$ (or $\frac{\pi}{b}$ for tangent), phase shift $= h$, midline $y = k$.
- **Always factor out b first** before reading the phase shift.
- $\arcsin()$, $\arccos()$, $\arctan()$ invert the trig functions on their restricted domains. Outputs are angles, not positional values.
- Trig equations have infinitely many solutions. Obtain the general solution by adding $+2\pi k$ (for sine and cosine) or $+\pi k$ (for tangent) once you've applied the inverse trigonometric functions.

Looking Ahead

- Next class is our final new content day — we apply trigonometry to right triangles and see how $\sin()$ and $\cos()$ connect to the ratios of sides that motivated the subject at the start of the unit.
- This is also where we return to the projectile problem from Day 23 and finish what we started.

Next Time:

Right Triangle Trigonometry and Applications

Homework:

Continue Homework 12 on MyOpenMath