

MAT 142: Exponential and Logarithm Foundations

Dr. Gilbert

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Reminders

Last class we studied radical functions — the final class of algebraic function in our course. Today we begin an investigation of two genuinely new function classes.

Try the following warm-up problems.

Problem 1: Which of the following are radical functions? Which are rational functions? Which are neither?

$$f(x) = \frac{x^2 - 5x}{x^2 - 25}, \quad g(x) = 3\sqrt[5]{(x^2 - 9x + 20)}, \quad h(x) = \frac{x + 8}{\sqrt{(x^2 - 36)}}$$

Problem 2: Find the domain of each function in Problem 1.

Problem 3: A student claims that $\sqrt{(x^2)} = x$ for all real x . Is the student correct? If not, give a counterexample and write the correct statement.

Objectives

Today we review two genuinely new classes of function that behave differently from everything we've seen so far: *exponential* and *logarithmic* functions.

After today's class meeting, you should be able to:

- Apply the laws of exponents to simplify exponential expressions.
- Convert between logarithmic and exponential form.
- Apply the laws of logarithms to expand and condense logarithmic expressions.
- Use inverse properties of exponentials and logarithms to simplify expressions.
- Solve exponential and logarithmic equations.

Laws of Exponents

The following properties govern all exponential expressions of the form a^b .

$$(1) x^0 = 1, \text{ for } x \neq 0 \quad (2) x^1 = x$$

$$(3) x^m x^n = x^{m+n} \quad (4) \frac{x^m}{x^n} = x^{m-n}$$

$$(5) (x^a)^b = x^{ab} \quad (6) x^{-a} = \frac{1}{x^a}, \text{ for } x \neq 0$$

$$(7) (xy)^a = x^a y^a \quad (8) \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

These are the same rules you used in algebra (and we reviewed in our Week One class meetings).

They extend unchanged to any real exponent, including fractions, negatives, and variables.

Simplifying Exponential Expressions

Example 1: Simplify each expression. Write your answer with only positive exponents.

$$(a) \ (x^2 y^{-3})^2 x y^5 \qquad (b) \ \left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} \qquad (c) \ \left(\frac{x^8 y^2}{(xy)^2} \right)^{1/2}$$

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(a)

$$(x^2 y^{-3})^2 xy^5$$

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(a)

$$(x^2y^{-3})^2xy^5 = x^4y^{-6}xy^5$$

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(a)

$$\begin{aligned} (x^2 y^{-3})^2 xy^5 &= x^4 y^{-6} xy^5 \\ &= (x^4 x) (y^{-6} y^5) \end{aligned}$$

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$$\begin{aligned}(x^2 y^{-3})^2 xy^5 &= x^4 y^{-6} xy^5 \\ &= (x^4 x) (y^{-6} y^5) \\ &= x^5 y^{-1}\end{aligned}$$

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(a)

$$\begin{aligned} (x^2 y^{-3})^2 x y^5 &= x^4 y^{-6} x y^5 \\ &= (x^4 x) (y^{-6} y^5) \\ &= x^5 y^{-1} \\ &= \boxed{\frac{x^5}{y}} \end{aligned}$$

(b)

$$\left(\frac{xy^2}{x^{-4}y^6} \right)^{-2}$$

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(b)

$$\left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} = \frac{x^{-2}y^{-4}}{x^8 y^{-12}}$$

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(b)

$$\begin{aligned} \left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8 y^{-12}} \\ &= \frac{y^{12}}{x^2 y^4 x^8} \end{aligned}$$

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$$\begin{aligned}\left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8 y^{-12}} \\&= \frac{y^{12}}{x^2 y^4 x^8} \\&= \frac{y^{12}}{(x^2 x^8) y^4}\end{aligned}$$

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$$\begin{aligned}\left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8 y^{-12}} \\&= \frac{y^{12}}{x^2 y^4 x^8} \\&= \frac{y^{12}}{(x^2 x^8) y^4} \\&= \frac{y^{12}}{x^{10} y^4}\end{aligned}$$

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$$\begin{aligned} \left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8y^{-12}} \\ &= \frac{y^{12}}{x^2y^4x^8} \\ &= \frac{y^{12}}{(x^2x^8)y^4} \\ &= \frac{y^{12}}{x^{10}y^4} \\ &= \boxed{\frac{y^8}{x^{10}}} \end{aligned}$$

(c)

$$\left(\frac{x^8y^2}{(xy)^2}\right)^{1/2}$$

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$$\begin{aligned}\left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8 y^{-12}} \\&= \frac{y^{12}}{x^2 y^4 x^8} \\&= \frac{y^{12}}{(x^2 x^8) y^4} \\&= \frac{y^{12}}{x^{10} y^4} \\&= \boxed{\frac{y^8}{x^{10}}}\end{aligned}$$

(c)

$$\left(\frac{x^8 y^2}{(xy)^2} \right)^{1/2} = \frac{x^4 y}{xy}$$

Simplifying Exponential Expressions

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$$\begin{aligned}(x^2 y^{-3})^2 x y^5 &= x^4 y^{-6} x y^5 \\&= (x^4 x) (y^{-6} y^5) \\&= x^5 y^{-1} \\&= \boxed{\frac{x^5}{y}}\end{aligned}$$

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$$\begin{aligned}\left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8 y^{-12}} \\&= \frac{y^{12}}{x^2 y^4 x^8} \\&= \frac{y^{12}}{(x^2 x^8) y^4} \\&= \frac{y^{12}}{x^{10} y^4} \\&= \boxed{\frac{y^8}{x^{10}}}\end{aligned}$$

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$$\begin{aligned}\left(\frac{x^8 y^2}{(xy)^2} \right)^{1/2} &= \frac{x^4 y}{xy} \\&= \frac{x^3 y}{y}\end{aligned}$$

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(b)

$$\begin{aligned} \left(\frac{xy^2}{x^{-4}y^6} \right)^{-2} &= \frac{x^{-2}y^{-4}}{x^8y^{-12}} \\ &= \frac{y^{12}}{x^2y^4x^8} \\ &= \frac{y^{12}}{(x^2x^8)y^4} \\ &= \frac{y^{12}}{x^{10}y^4} \\ &= \boxed{\frac{y^8}{x^{10}}} \end{aligned}$$

(c)

$$\begin{aligned} \left(\frac{x^8 y^2}{(xy)^2} \right)^{1/2} &= \frac{x^4 y}{xy} \\ &= \frac{x^3 y}{y} \\ &= \boxed{x^3} \end{aligned}$$

Simplifying Exponential Expressions Practice

Try It! 1: Simplify $\left(\frac{x^3y^{-2}}{x^{-1}y^4}\right)^2$. Write your answer with only positive exponents.

Try It! 2: Simplify $\frac{(x^2y)^3 \cdot x^{-4}}{y^{-2} \cdot (xy^2)^2}$. Write your answer using only positive exponents.

Try It! 3: Simplify $(x^{-1/2}y^2)^4 \cdot (x^3y^{-1})^{-2}$. Write your answer using only positive exponents.

What Is a Logarithm?

Definition: The *logarithm base b of a* is the exponent to which we must raise b to obtain a :

$$\log_b(a) = y \iff b^y = a \quad \text{or} \quad \log_{\square}(\triangle) = \bigcirc \iff \square^{\bigcirc} = \triangle$$

Logarithmic and exponential statements contain the same information, just expressed differently.

Reading the definition: The *base* of the logarithm is the base of the corresponding exponential. The *argument* is the result of the exponentiation. The *output* of the logarithm is the required exponent.

Domain of logarithms: Since a positive base raised to any real power is always positive, $\log_b(a)$ is only defined for $a > 0$. The domain of any logarithmic function is $(0, \infty)$.

Put another way, we cannot take logarithms of negative numbers (or of 0).

Evaluating Logarithms

Example 2: Evaluate each logarithm by first converting to exponential form.

(1) $\log_2(8)$

Ask: For what y is $2^y = 8$?

Since $2^3 = 8$, we have $\log_2(8) = 3$.

(2) $\log_3(81)$

Ask: For what y is $3^y = 81$?

Since $3^4 = 81$, we have $\log_3(81) = 4$.

(3) $\log_4\left(\frac{1}{64}\right)$

Ask: For what y is $4^y = \frac{1}{64}$?

We know that the exponent must be negative.
Compensating for this, ask for what z is $4^z = 64$?

Since $4^3 = 64$, we have $\log_4\left(\frac{1}{64}\right) = -3$.

(4) $\log_{1/9}(81)$

Ask: For what y is $\left(\frac{1}{9}\right)^y = 81$?

We know that the exponent must be negative.
Compensating for that, ask for what z is $9^z = 81$?

Since $9^2 = 81$, we know that $\log_{1/9}(81) = -2$.

Laws of Logarithms

The following properties govern logarithmic expressions. Each one is a direct consequence of the corresponding law of exponents.

$$(1) \log_a(1) = 0$$

$$(2) \log_a(a) = 1$$

$$(3) \log_a(xy) = \log_a(x) + \log_a(y) \quad (4) \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$(5) \log_a(x^n) = n \log_a(x) \quad (6) \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \quad (\text{Change of Base})$$

Two special bases:

- $\log_{10}(x)$ is written simply as $\log(x)$ — the *common logarithm*.
- $\log_e(x)$ is written as $\ln(x)$ — the *natural logarithm*, with base $e \approx 2.718$.

Simplifying Logarithmic Expressions

Example 3: Use the laws of logarithms to write each expression as a single logarithm.

$$(a) \ 2 \log_2(x) - \log_2(5x) \qquad (b) \ \ln(x) + \ln(x+1) - \ln(x^2) \qquad (c) \ \log_{10}(2x) + \log_{10}\left(\frac{5}{x}\right)$$

(a)

$$2 \log_2(x) - \log_2(5x)$$

Simplifying Logarithmic Expressions

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(a)

$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \end{aligned}$$

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(a)

$$\begin{aligned} & 2\log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \end{aligned}$$

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(a)

$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \\ &= \boxed{\log_2\left(\frac{x}{5}\right)} \end{aligned}$$

(b)

$$\ln(x) + \ln(x+1) - \ln(x^2)$$

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$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \\ &= \boxed{\log_2\left(\frac{x}{5}\right)} \end{aligned}$$

(b)

$$\begin{aligned} & \ln(x) + \ln(x + 1) - \ln(x^2) \\ &= \ln(x(x + 1)) - \ln(x^2) \end{aligned}$$

Simplifying Logarithmic Expressions

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$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \\ &= \boxed{\log_2\left(\frac{x}{5}\right)} \end{aligned}$$

(b)

$$\begin{aligned} & \ln(x) + \ln(x + 1) - \ln(x^2) \\ &= \ln(x(x + 1)) - \ln(x^2) \\ &= \ln\left(\frac{x(x + 1)}{x^2}\right) \end{aligned}$$

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(a) (b) (c)

$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \\ &= \boxed{\log_2\left(\frac{x}{5}\right)} \end{aligned}$$

$$\begin{aligned} & \ln(x) + \ln(x + 1) - \ln(x^2) \\ &= \ln(x(x + 1)) - \ln(x^2) \\ &= \ln\left(\frac{x(x + 1)}{x^2}\right) \\ &= \boxed{\ln\left(\frac{x + 1}{x}\right)} \end{aligned}$$

$$\log_{10}(2x) + \log_{10}\left(\frac{5}{x}\right)$$

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(b)

$$\begin{aligned} & \ln(x) + \ln(x + 1) - \ln(x^2) \\ &= \ln(x(x + 1)) - \ln(x^2) \\ &= \ln\left(\frac{x(x + 1)}{x^2}\right) \\ &= \boxed{\ln\left(\frac{x + 1}{x}\right)} \end{aligned}$$

(c)

$$\begin{aligned} & \log_{10}(2x) + \log_{10}\left(\frac{5}{x}\right) \\ &= \log_{10}\left(2x \left(\frac{5}{x}\right)\right) \end{aligned}$$

Simplifying Logarithmic Expressions

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$$(a) \ 2 \log_2(x) - \log_2(5x) \qquad (b) \ \ln(x) + \ln(x+1) - \ln(x^2) \qquad (c) \ \log_{10}(2x) + \log_{10}\left(\frac{5}{x}\right)$$

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$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \\ &= \boxed{\log_2\left(\frac{x}{5}\right)} \end{aligned}$$

(b)

$$\begin{aligned} & \ln(x) + \ln(x+1) - \ln(x^2) \\ &= \ln(x(x+1)) - \ln(x^2) \\ &= \ln\left(\frac{x(x+1)}{x^2}\right) \\ &= \boxed{\ln\left(\frac{x+1}{x}\right)} \end{aligned}$$

(c)

$$\begin{aligned} & \log_{10}(2x) + \log_{10}\left(\frac{5}{x}\right) \\ &= \log_{10}\left(2x \left(\frac{5}{x}\right)\right) \\ &= \log_{10}(10) \end{aligned}$$

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$$\begin{aligned} & 2 \log_2(x) - \log_2(5x) \\ &= \log_2(x^2) - \log_2(5x) \\ &= \log_2\left(\frac{x^2}{5x}\right) \\ &= \boxed{\log_2\left(\frac{x}{5}\right)} \end{aligned}$$

$$\begin{aligned} & \ln(x) + \ln(x+1) - \ln(x^2) \\ &= \ln(x(x+1)) - \ln(x^2) \\ &= \ln\left(\frac{x(x+1)}{x^2}\right) \\ &= \boxed{\ln\left(\frac{x+1}{x}\right)} \end{aligned}$$

$$\begin{aligned} & \log_{10}(2x) + \log_{10}\left(\frac{5}{x}\right) \\ &= \log_{10}\left(2x \left(\frac{5}{x}\right)\right) \\ &= \log_{10}(10) \\ &= \boxed{1} \end{aligned}$$

Simplifying Logarithmic Expressions Practice

Try It! 4: Evaluate $\log_8(4)$ by converting to exponential form. (*Hint: write both 8 and 4 as powers of 2.*)

Try It! 5: Write $3\ln(x) + \ln(x^2 + 1) - 2\ln(x - 1)$ as a single logarithm.

Try It! 6: Write $\log_2(x + 3) + 3\log_2(x) - \log_2(x^2 - 9)$ as a single logarithm.

Try It! 7: Write $\frac{1}{2}\ln(x^2 + 4) - 2\ln(x) + \ln(x^3)$ as a single logarithm.

Inverse Properties

Since $f(x) = a^x$ and $g(x) = \log_a(x)$ are inverses of each other, composing them in either order returns x :

$$a^{\log_a(x)} = x \quad \text{and} \quad \log_a(a^x) = x$$

These two identities are powerful simplification tools.

Example 4: Simplify each expression.

(a) $3 \log_{10}(10^2) + 5$

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(a) $3 \log_{10}(10^2) + 5$

$$3 \log_{10}(10^2) + 5 = 3(2) + 5$$

Inverse Properties

Since $f(x) = a^x$ and $g(x) = \log_a(x)$ are inverses of each other, composing them in either order returns x :

$$a^{\log_a(x)} = x \quad \text{and} \quad \log_a(a^x) = x$$

These two identities are powerful simplification tools.

Example 4: Simplify each expression.

(a) $3 \log_{10}(10^2) + 5$

$$\begin{aligned} 3 \log_{10}(10^2) + 5 &= 3(2) + 5 \\ &= \boxed{11} \end{aligned}$$

(b) $10^{(\log_{10}(4) + \log_{10}(15)) - \log_{10}(3)}$

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Solving Exponential Equations

An *exponential equation* has the variable in an exponent.

Since our basic operations (addition, subtraction, multiplication, division, roots, and powers) can't reach it there, we use logarithms to bring the variable down.

Strategy:

1. Isolate the exponential expression on one side.
2. Take **ln** of both sides and use $\ln(a^n) = n \ln(a)$ to bring the exponent down.
3. Solve for the variable using standard algebra.

Choice of Logarithm Base

In Step 2 above, any base works, but **ln** is a perfectly convenient choice. Sometimes a more appropriate choice of base can make your work a bit simpler, but you'll always arrive at the same solution.

Solving an Exponential Equation I

Example 5: Solve $5(4^{2x+7}) - 12 = -2$.

Solution. Isolate the exponential first.

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$$5(4^{2x+7}) - 12 = -2$$

$$\implies 5(4^{2x+7}) = 10$$

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Solving an Exponential Equation I

Example 5: Solve $5(4^{2x+7}) - 12 = -2$.

Solution. Isolate the exponential first. Next take the natural logarithm of both sides to bring the exponent down.

$$5(4^{2x+7}) - 12 = -2$$

$$\implies 5(4^{2x+7}) = 10$$

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$$\implies 5(4^{2x+7}) = 10$$

$$\implies 4^{2x+7} = 2$$

$$\implies \ln(4^{2x+7}) = \ln(2)$$

Solving an Exponential Equation I

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Solution. Isolate the exponential first. Next take the natural logarithm of both sides to bring the exponent down.

$$\begin{aligned}5(4^{2x+7}) - 12 &= -2 \\ \implies 5(4^{2x+7}) &= 10 \\ \implies 4^{2x+7} &= 2 \\ \implies \ln(4^{2x+7}) &= \ln(2) \\ \implies (2x + 7) \ln(4) &= \ln(2)\end{aligned}$$

Solving an Exponential Equation I

Example 5: Solve $5(4^{2x+7}) - 12 = -2$.

Solution. Isolate the exponential first. Next take the natural logarithm of both sides to bring the exponent down. Use algebra to isolate the variable.

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Using the change of base rule gives us

$$x = \frac{\log_4(2) - 7}{2}.$$

We know that $\log_4(2)$ is $\frac{1}{2}$ because $4^{1/2} = 2$.

This gives us $x = \frac{\frac{1}{2} - 7}{2}$

Finding a common denominator and simplifying

further gives us $x = \frac{-13}{4}$.

Solving an Exponential Equation Practice

Try It! 8: Solve $3^{x-2} = 15$. Leave your answer in exact form involving the natural logarithm.

Try It! 9: Solve $3^{x^2-2x} = 3^{x+4}$.

Try It! 10: Solve $5^{4-x} = 30$.

Try It! 11: Solve $3(7^{3x-1}) + 5 = 26$.

Solving Logarithmic Equations

A *logarithmic equation* has the variable inside a logarithm argument. We use exponentials to unlock it.

Strategy:

1. Isolate the logarithmic expression on one side.
2. Exponentiate both sides using the same base as the logarithm, exploiting $a^{\log_a(x)} = x$.
3. Solve for the variable using standard algebra.
4. **Check all solutions** — since the domain of a logarithm is $(0, \infty)$, some algebraically valid solutions may be extraneous.



Always Check Solutions

Extraneous solutions are common with logarithmic equations. A candidate solution is only valid if the argument to every logarithm in the original equation is strictly positive at that value.

Solving Logarithmic Equations I

Example 6: Solve $3 \log_5 (x^2 - 4x) + 7 = 10$.

Solution. Isolate the logarithm first.

$$3 \log_5 (x^2 - 4x) + 7 = 10$$

Solving Logarithmic Equations I

Example 6: Solve $3 \log_5 (x^2 - 4x) + 7 = 10$.

Solution. Isolate the logarithm first.

$$\begin{aligned} 3 \log_5 (x^2 - 4x) + 7 &= 10 \\ \implies 3 \log_5 (x^2 - 4x) &= 3 \end{aligned}$$

Solving Logarithmic Equations I

Example 6: Solve $3 \log_5 (x^2 - 4x) + 7 = 10$.

Solution. Isolate the logarithm first.

$$3 \log_5 (x^2 - 4x) + 7 = 10$$

$$\implies 3 \log_5 (x^2 - 4x) = 3$$

$$\implies \log_5 (x^2 - 4x) = 1$$

Solving Logarithmic Equations I

Example 6: Solve $3 \log_5 (x^2 - 4x) + 7 = 10$.

Solution. Isolate the logarithm first. Next, exponentiate both sides with the appropriate base.

$$\begin{aligned} 3 \log_5 (x^2 - 4x) + 7 &= 10 \\ \implies 3 \log_5 (x^2 - 4x) &= 3 \\ \implies \log_5 (x^2 - 4x) &= 1 \end{aligned}$$

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Solving Logarithmic Equations I

Example 6: Solve $3 \log_5 (x^2 - 4x) + 7 = 10$.

Solution. Isolate the logarithm first. Next, exponentiate both sides with the appropriate base. Use algebraic techniques to solve what remains.

$$\begin{aligned} 3 \log_5 (x^2 - 4x) + 7 &= 10 \\ \implies 3 \log_5 (x^2 - 4x) &= 3 \\ \implies \log_5 (x^2 - 4x) &= 1 \\ \implies 5^{\log_5 (x^2 - 4x)} &= 5^1 \\ \implies x^2 - 4x &= 5 \end{aligned}$$

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Solving Logarithmic Equations I

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Solution. Isolate the logarithm first. Next, exponentiate both sides with the appropriate base. Use algebraic techniques to solve what remains.

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$$\implies \log_5 (x^2 - 4x) = 1$$

$$\implies 5^{\log_5 (x^2 - 4x)} = 5^1$$

$$\implies x^2 - 4x = 5$$

$$\implies x^2 - 4x - 5 = 0$$

$$\implies (x - 5)(x + 1) = 0$$

Solving Logarithmic Equations I

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Solving Logarithmic Equations I

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Now we'll verify whether the candidate solutions above are true solutions or are extraneous.

Doing so requires determining whether $x^2 - 4x$ is positive for each candidate solution.

Since $(-1)^2 - 4(-1) = 5$, we know that $\boxed{x = -1}$ is a valid solution.

Similarly, since $(5)^2 - 4(5) = 5$, we have $\boxed{x = 5}$ is also a valid solution.

Solving Logarithmic Equations Practice

Try It! 12: Solve $\log_3(x + 6) + \log_3(x) = 3$. Check your solutions carefully.

Try It! 13: Solve $\log_2(x) + \log_2(x - 2) = 3$. Check your solutions carefully.

Try It! 14: Solve $\log_4(x + 3) - \log_4(x) = \log_4(x - 1)$. Check your solutions carefully.

Try It! 15: Solve $2 \log_3(x) - \log_3(x + 6) = 1$. Check your solutions carefully.

Exit Ticket Task

Navigate to [our MAT142 Exit Ticket Form](#), answer the questions, and complete the *task* below.

Note. Today's discussion is listed as [16. Exponential and Logarithm Foundations](#)



Task: Consider the equation $2\log_4(x - 1) = 2$.

- (a) Isolate the logarithm and exponentiate both sides.
- (b) Solve for x .
- (c) Check whether your solution is valid.

Summary and Next Time...

Ideas From Today

- The **laws of exponents** govern all exponential expressions: multiply $\rightarrow$ add exponents, divide $\rightarrow$ subtract, power of power $\rightarrow$ multiply, negative exponent $\rightarrow$ reciprocal.
- $\log_b(a) = y \iff b^y = a$. Logarithms and exponentials are two representations of the same relationship.
- The **laws of logarithms** mirror the laws of exponents: products $\rightarrow$ sums, quotients $\rightarrow$ differences, powers $\rightarrow$ coefficients.
- The **inverse properties** $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$ are the key tools for solving equations.
- Solving **exponential equations**: isolate the exponential and then take $\ln()$ of both sides.
- Solving **logarithmic equations**: isolate the logarithm and then exponentiate both sides. Always check for extraneous solutions.

Looking Ahead

- Today was a review / overview of mechanics. Next class we look at exponential and logarithmic **functions** — their graphs, transformations, and behavior — and see why these functions appear throughout science, finance, and data analysis.

Next Time:

Exponential and Logarithmic Functions

Homework:

Start Homework 11 on MyOpenMath