MAT 142: Rational Functions and Equations

Dr. Gilbert

June 8, 2026

Reminders

At our last meeting, we encountered and utilized two tools for working with polynomials in expanded form:

- The **Rational Roots Theorem** narrowed down candidates for rational roots.
- **Polynomial long division** was used to factor out a known root r of $p\left(x\right)$, rewriting $p\left(x\right)=\left(x-r\right)q\left(x\right)$ where $q\left(x\right)$ is of lower degree than $p\left(x\right)$.
- Together, these tools let us factor any polynomial into *linear* and *quadratic* factors, ultimately permitting us to find all of its roots.

Try the following warm-up problems.

Problem 1: Use polynomial long division to factor $p\left(x\right)=x^3+x^2-21x-45$ and sketch a graph. *(Hint:* $(x+3)$ *is a factor.)*

Problem 2: Find the domain of $f\left(x\right)=\dfrac{x^2-5x}{x^2-25}$.

Problem 3: Find the domain of $f\left(x\right)=\dfrac{x+8}{\sqrt{(x^2-36)}}$.

Note: Problems 2 and 3 both involve domain restrictions, but for different reasons. Think about what makes each one different.

Objectives

Today we'll leverage those tools from polynomial functions to analyse *rational functions*, which are ratios of polynomials.

Rational functions introduce genuinely new geometric features that our previous classes of functions don't have.

- Domain restrictions
- Asymptotes

After today's class meeting, you should be able to:

- Define rational functions and identify their domains.
- Find the roots and y-intercept of a rational function.
- Identify vertical and horizontal asymptotes and explain what they represent.
- Use asymptotes, roots, the y-intercept, and additional reference points to sketch the graph of a rational function.

What Is a Rational Function?

Definition (Rational Function): A function $f(x)$ is a *rational function* if it can be written as

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

A few notes:

- Polynomial functions are rational functions with $q(x) = 1$.
- Rational functions introduce a new concern in that the denominator can possibly evaluate to zero, creating *domain restrictions* and *asymptotes*.
- Finding roots of a rational function means solving $\frac{p(x)}{q(x)} = 0$, which is a *rational equation*.

Domain, Roots, and y-Intercept

Consider the general rational function $r(x) = \dfrac{p(x)}{q(x)}$

Domain: Since division by zero is undefined, the domain of $r(x)$ excludes all x where $q(x) = 0$.

Roots: A fraction equals zero only when its *numerator* is zero (and its denominator is not). So roots occur where $p(x) = 0$, provided $q(x) \neq 0$ at those points.

y-Intercept: Evaluate $r(0)$, provided 0 is in the domain. If $x = 0$ is not a permissible input, then there is no y-intercept.

Analysing Rational Functions I

Example 1: For $r\left(x\right)=\dfrac{2x^2-8x+6}{x^2-x-20}$, find the domain, roots, and y-intercept.

Solution.

Domain: Solve $x^2-x-20\neq 0$.

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So the domain is $x \neq -4$ and $x \neq 5$, or

$\boxed{(-\infty, -4) \cup (-4, 5) \cup (5, \infty)}$.

Roots: Solve $\dfrac{2x^2 - 8x + 6}{x^2 - x - 20} = 0$

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$$\begin{aligned} \frac{2x^2 - 8x + 6}{x^2 - x - 20} &= 0 \\ \implies 2x^2 - 8x + 6 &= 0 \\ \implies 2\left(x^2 - 4x + 3\right) &= 0 \end{aligned}$$

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y-intecept: Since $x=0$ is in the domain of $r\left(x\right)$, the function has a y-intercept.

We'll find it by evaluating $r\left(0\right)$.

$$r\left(0\right)=\frac{2(0)^{2}-8\left(0\right)+6}{(0)^{2}-\left(0\right)-20}$$

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We can now draw a partial graph of the function $r\left(x\right)=\dfrac{2x^2-8x+6}{x^2-x-20}$.

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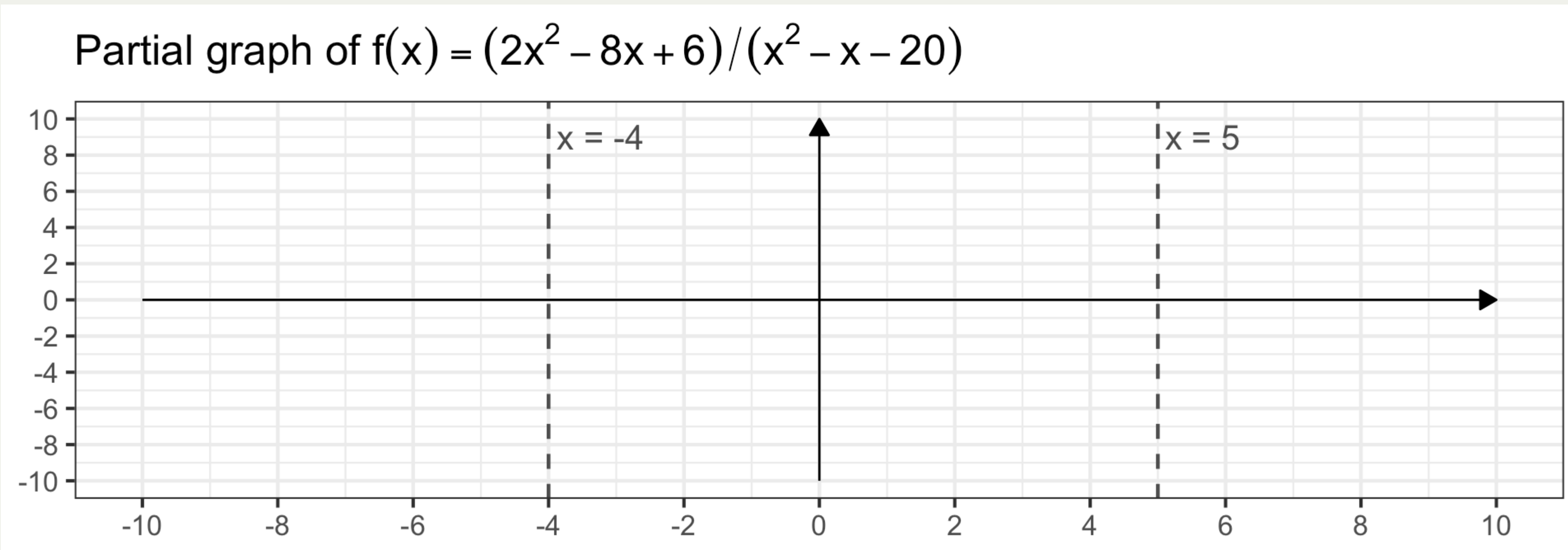

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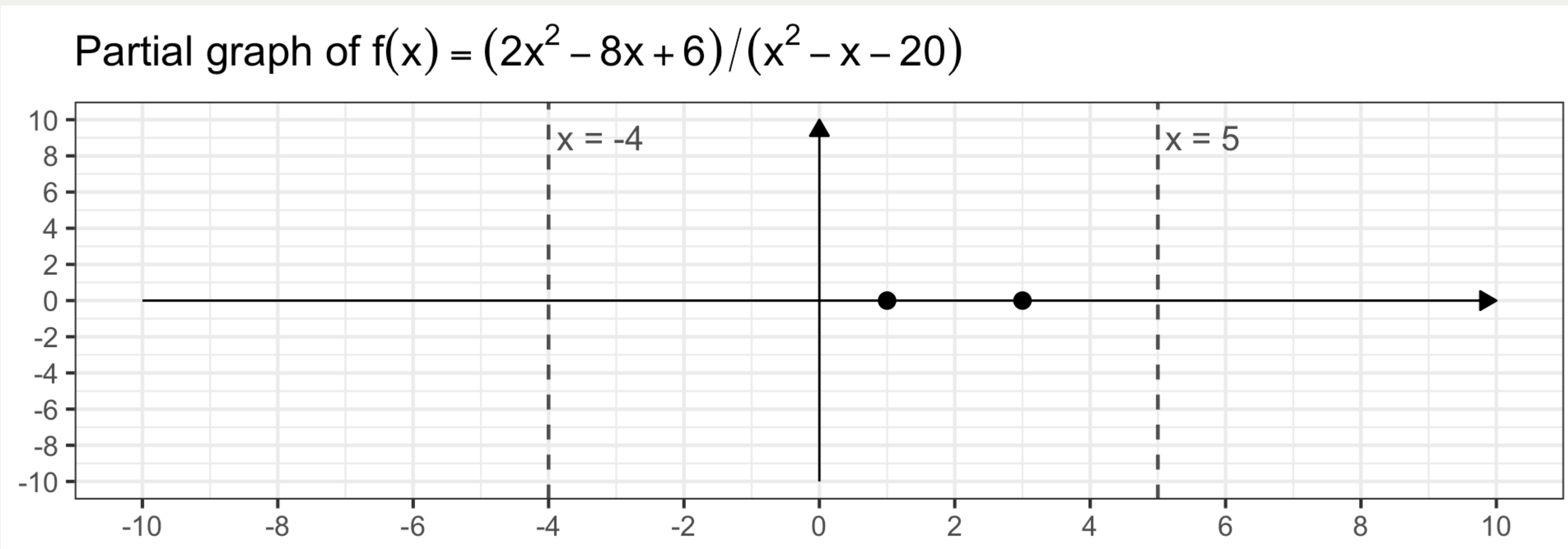

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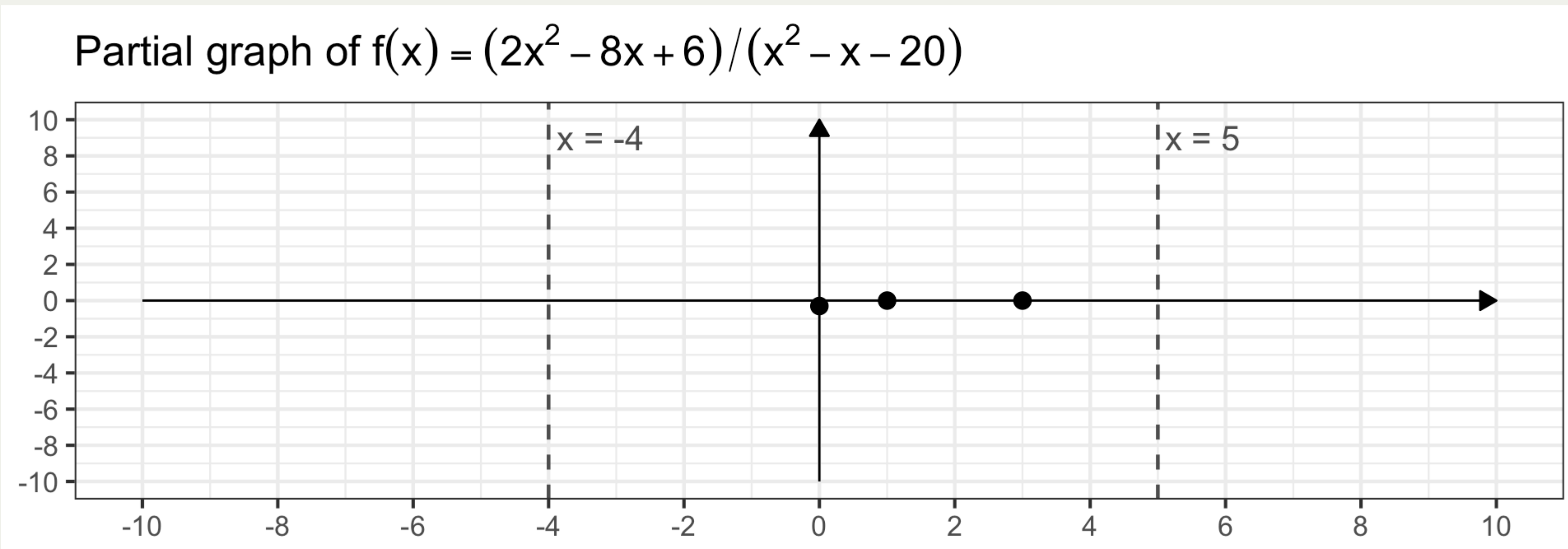

Analysis of Rational Function Practice

Try It! 1: Identify the domain, roots, and y-intercept for the function $g\left(x\right)=\dfrac{-4x^2-4x+24}{5x^2+5x-100}$. Draw a partial graph of the function by identifying locations for discontinuities, the y-intercept, and any roots.

Asymptotes

Definition (Asymptote): An asymptote is a line (or curve) that a function approaches arbitrarily closely. Rational functions can have *vertical*, *horizontal*, or *slant* asymptotes.

A Common Misconception

You may have previously heard that "*A function can never cross its asymptote*", but this is not generally true. A function can never cross its *vertical* asymptotes, but it *can* cross its horizontal or slant asymptote.

Definition (Vertical Asymptote): The line $x = a$ is a vertical asymptote of $\dfrac{p(x)}{q(x)}$ if $q(a) = 0$ and $p(a) \neq 0$.

Types of Discontinuities

There are two types of *discontinuity* that a rational function $r(x) = \dfrac{p(x)}{q(x)}$ may have.

- A *hole* occurs at $x = a$ if $(x - a)$ is a factor of both $p(x)$ and $q(x)$.
- A *vertical asymptote* occurs at $x = a$ when $(x - a)$ is a factor of $q(x)$, but not $p(x)$.

Vertical Asymptotes and Holes

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Strategy: Solve $q(x) = 0$. Each solution is either the location of a *vertical asymptote* or of a *hole*.

For $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$, we already found that $q(x) = 0$ at $x = -4$ and $x = 5$.

Since neither of these are also roots of $p(x)$, the discontinuities don't cancel out – they aren't *holes*.

Both of these discontinuities result in vertical asymptotes, represented by the dashed lines in the partial graph we produced earlier.

Vertical Asymptotes Practice

Try It! 2: Determine whether the domain restrictions of $g\left(x\right)=\dfrac{-4x^2-4x+24}{5x^2+5x-100}$ correspond to vertical asymptotes or holes.

Horizontal Asymptotes

Definition (Horizontal Asymptote): The line $y = b$ is a horizontal asymptote of $r(x)$ if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

Strategy: Divide numerator and denominator by x^n, where n is the highest degree in the entire expression. Then take the limit — terms with x in the denominator vanish.

Example 2: Find any horizontal asmyptotes of the function $f(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$.

*Solution.

$$\lim_{x \to -\infty} \frac{2x^2 - 8x + 6}{x^2 - x - 20}$$

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$$\lim_{x\to-\infty} \frac{2x^2 - 8x + 6}{x^2 - x - 20} = \lim_{x\to-\infty} \frac{\frac{2x^2}{x^2} - \frac{8x}{x^2} + \frac{6}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{20}{x^2}}$$

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\lim_{x\to-\infty} \frac{2x^2 - 8x + 6}{x^2 - x - 20} &= \lim_{x\to-\infty} \frac{\frac{2x^2}{x^2} - \frac{8x}{x^2} + \frac{6}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{20}{x^2}} \\
&= \lim_{x\to-\infty} \frac{2 - \frac{8}{x} + \frac{6}{x^2}}{1 - \frac{1}{x} - \frac{20}{x^2}}
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&= \frac{2}{1} \\
&= 2
\end{aligned}$$

So $r(x)$ has a horizontal asymptote at $y = 2$.

Note that the exact same analysis would show that $\lim_{x\to-\infty} r(x) = 2$.

If the end behavior of a rational function approaches a finite limit, then it will always be the case that $\lim_{x\to-\infty} r(x) = \lim_{x\to\infty} r(x)$.

This is not the case for all function classes though.

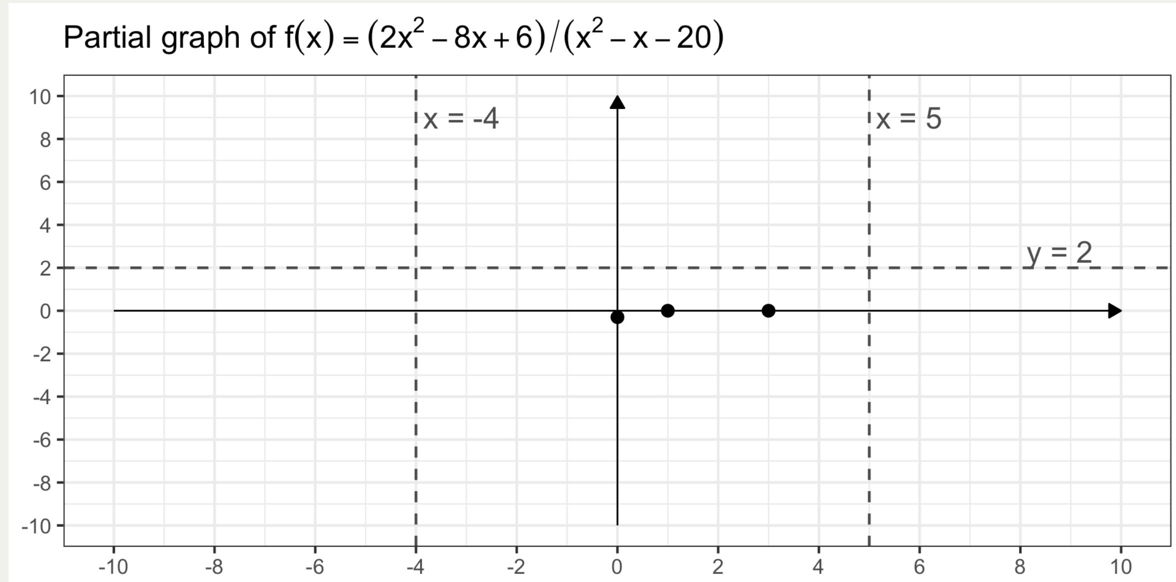
Horizontal Asymptotes Practice

Try It! 3: Determine whether the function $g\left(x\right) = \dfrac{-4x^2 - 4x + 24}{5x^2 + 5x - 100}$ has a horizontal asymptote. Write the equation of the horizontal asymptote, if so.

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



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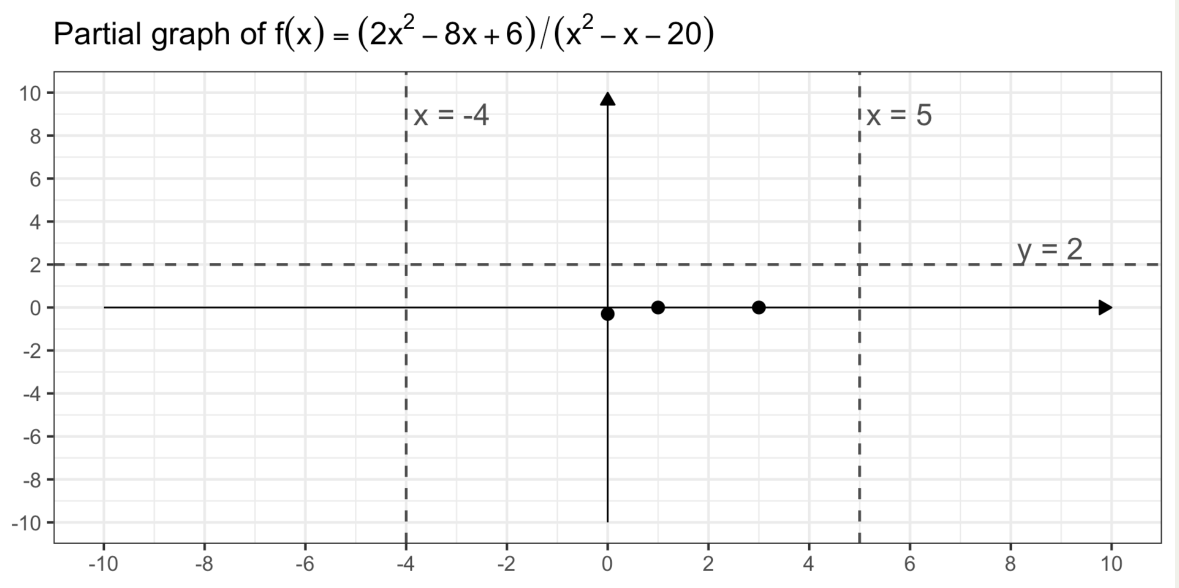
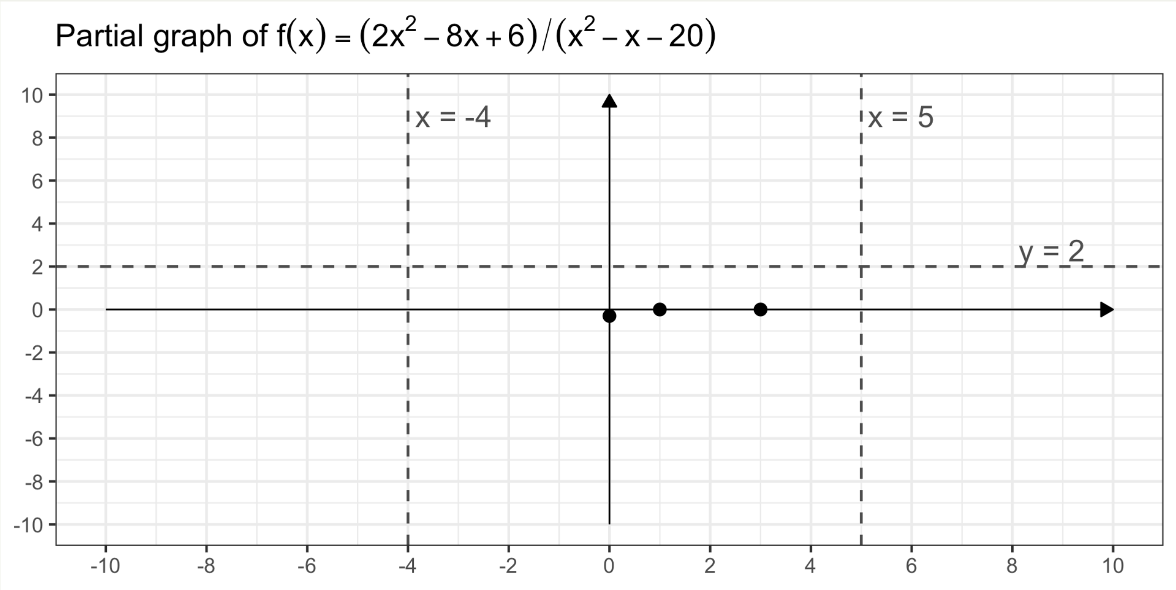


To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

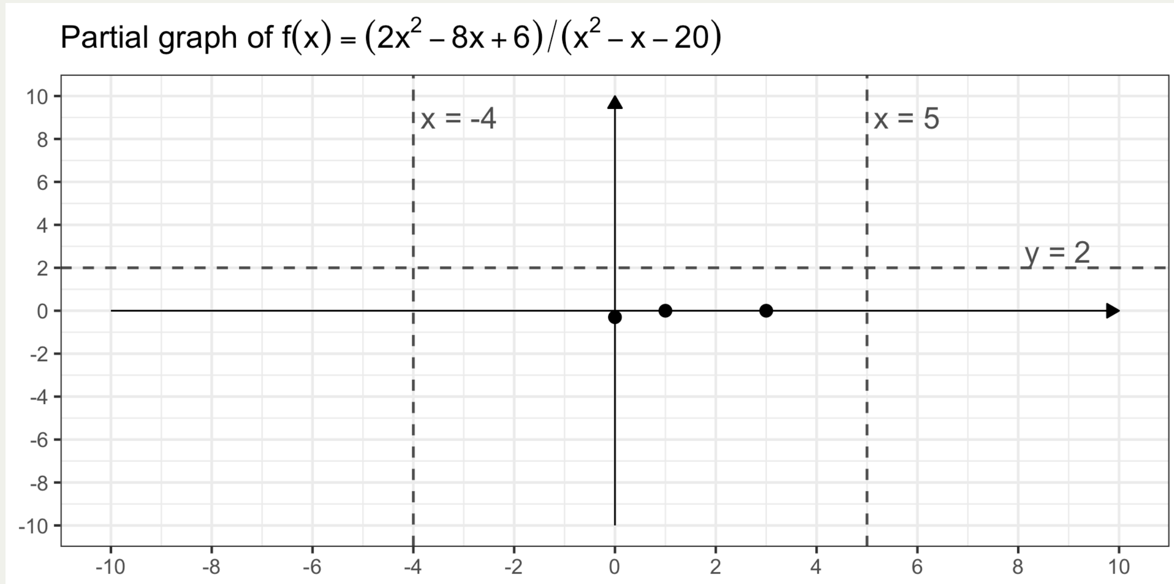
Reference Point at $x = -5$:

$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20}$$

Sketching the Complete Graph

We now know a great deal about $r\left(x\right)=\dfrac{2x^2-8x+6}{x^2-x-20}$. It has…

- vertical asymptotes at $x=-4$ and $x=5$,
- a horizontal asymptote at $y=2$,
- roots at $(1,0)$ and $(3,0)$, and
- a y-intercept at $(0,-3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x=-4$ and one to the right of $x=5$ because we already have reference points in the middle section.

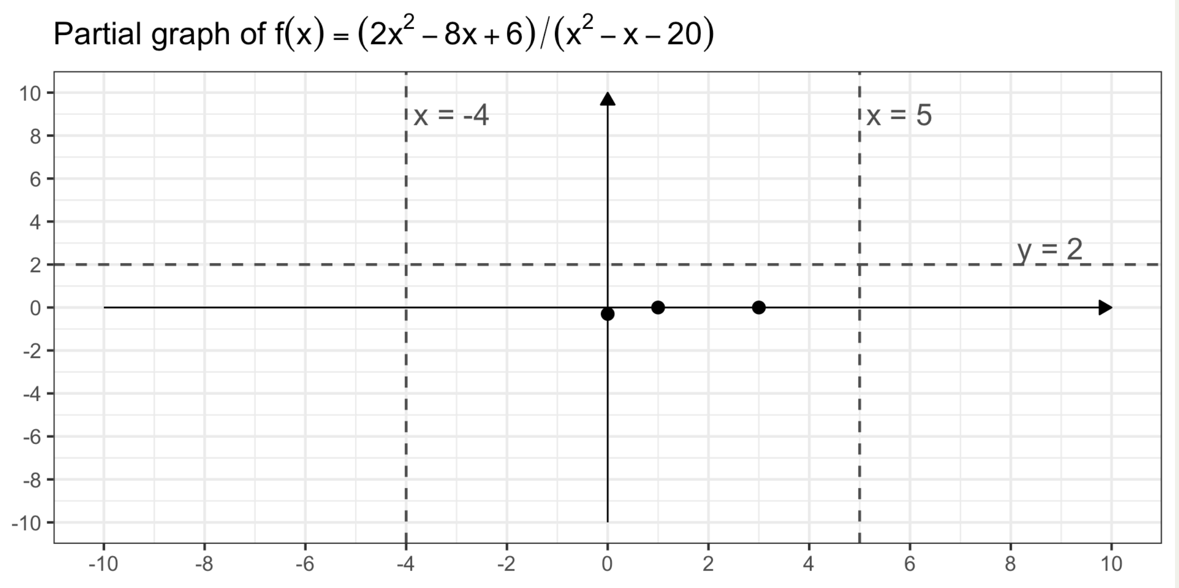
Reference Point at $x=-5$:

$$r\left(-5\right)=\frac{2(-5)^2-8\left(-5\right)+6}{(-5)^2-\left(-5\right)-20}=\frac{50+40+6}{25+5-20}$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

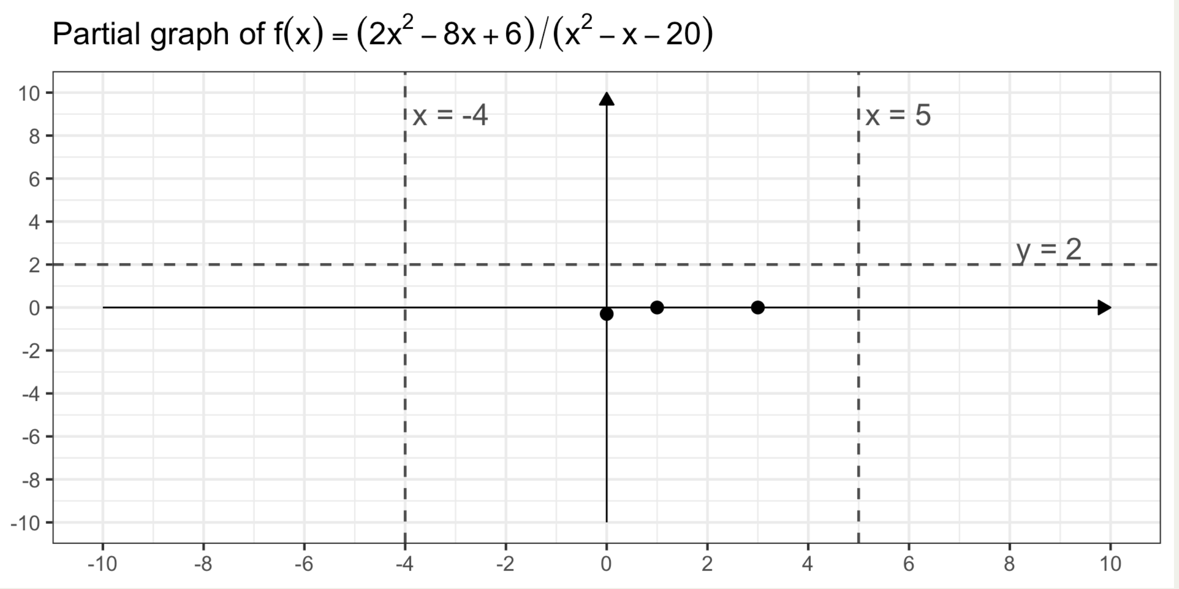
Reference Point at $x = -5$:

$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20} = \frac{96}{10}$$

Sketching the Complete Graph

We now know a great deal about $r\left(x\right)=\dfrac{2x^2-8x+6}{x^2-x-20}$. It has…

- vertical asymptotes at $x=-4$ and $x=5$,
- a horizontal asymptote at $y=2$,
- roots at $(1,0)$ and $(3,0)$, and
- a y-intercept at $(0,-3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x=-4$ and one to the right of $x=5$ because we already have reference points in the middle section.

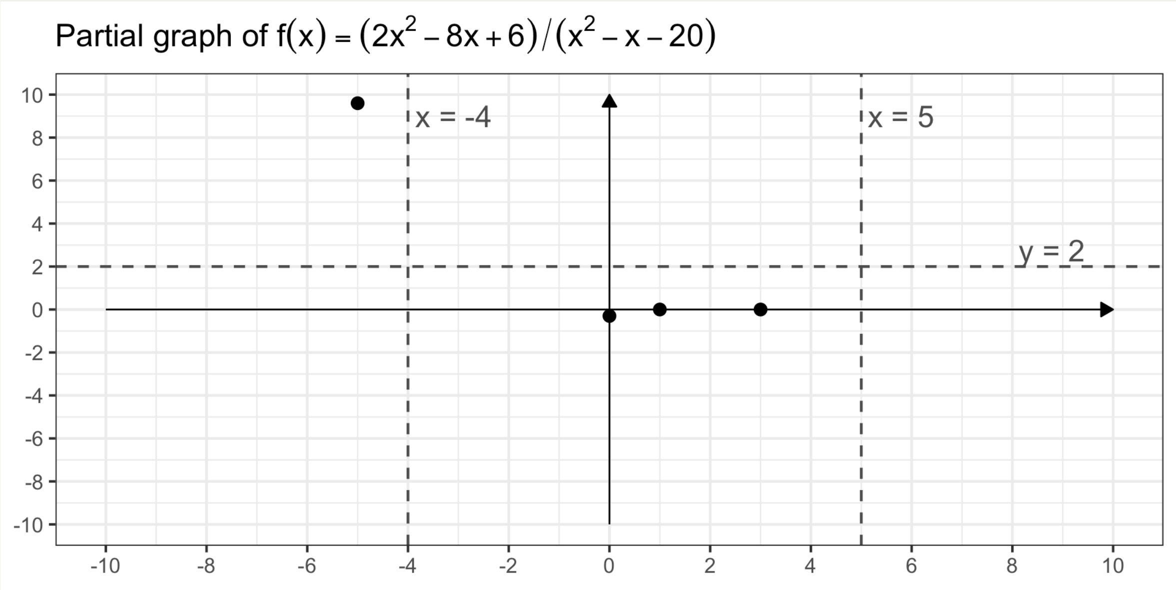
Reference Point at $x=-5$:

$$r\left(-5\right)=\frac{2(-5)^2-8\left(-5\right)+6}{(-5)^2-\left(-5\right)-20}=\frac{48}{5}$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

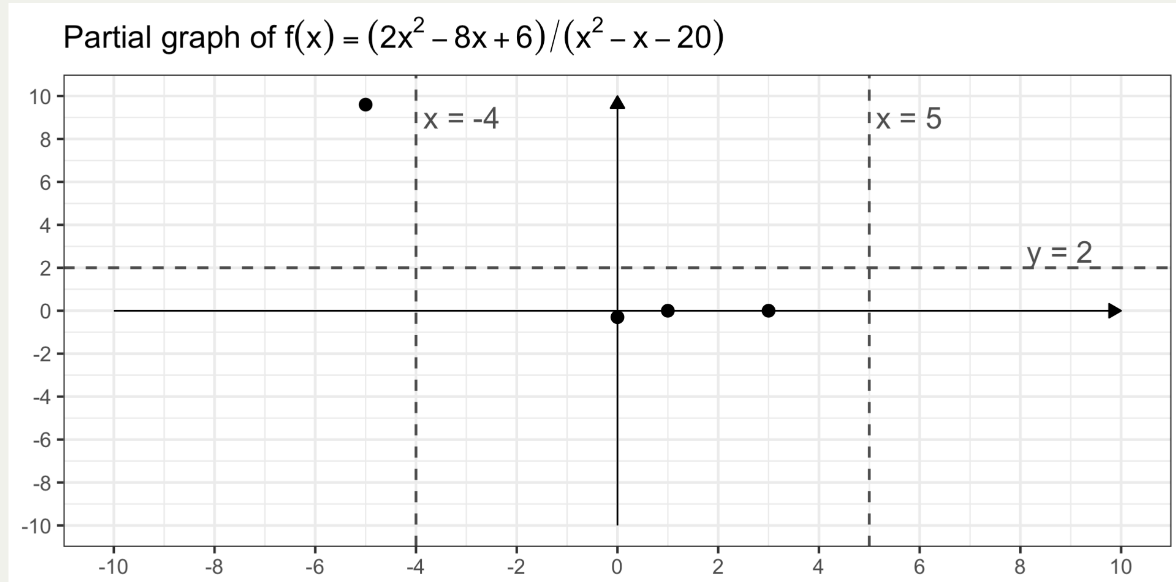
Reference Point at $x = -5$:

$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20} = \frac{48}{5} > 2$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

Reference Point at $x = -5$:

Reference Point at $x = 6$:

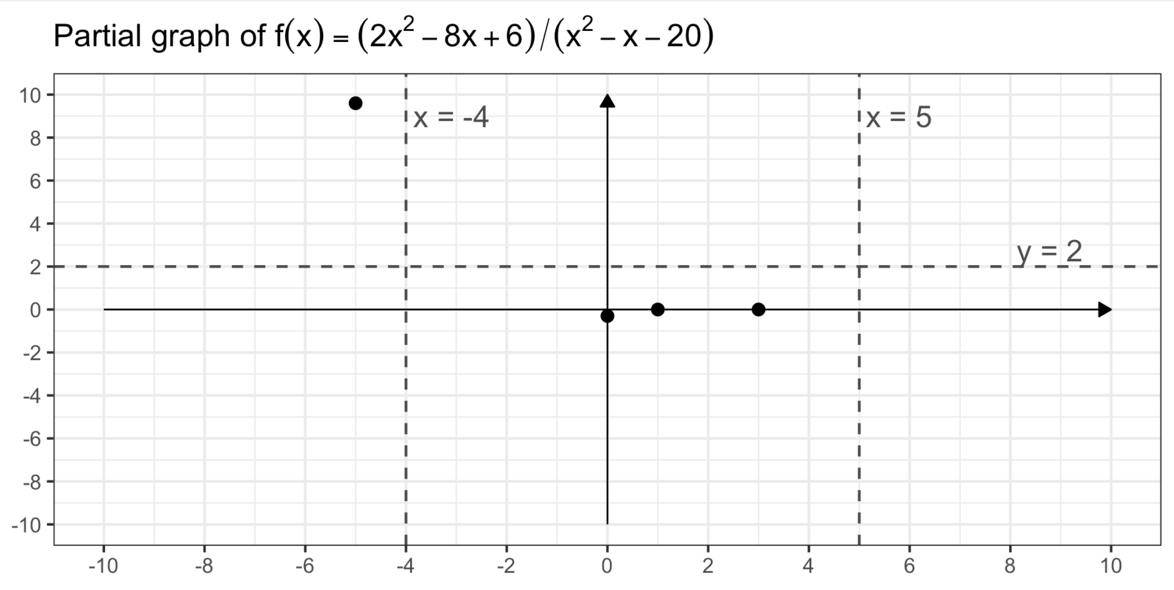
$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20} = \frac{48}{5} > 2$$

$$r(6) = \frac{2(6)^2 - 8(6) + 6}{(6)^2 - (6) - 20}$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

Reference Point at $x = -5$:

$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20} = \frac{48}{5} > 2$$

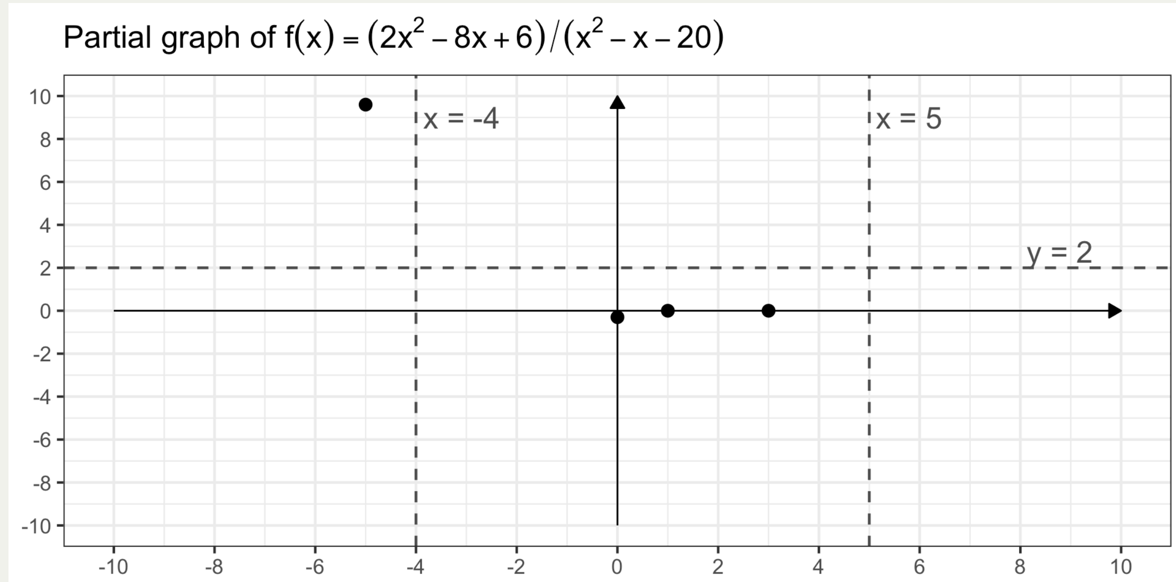
Reference Point at $x = 6$:

$$r(6) = \frac{2(6)^2 - 8(6) + 6}{(6)^2 - (6) - 20} = \frac{72 - 48 + 6}{36 - 6 - 20}$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

Reference Point at $x = -5$:

$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20} = \frac{48}{5} > 2$$

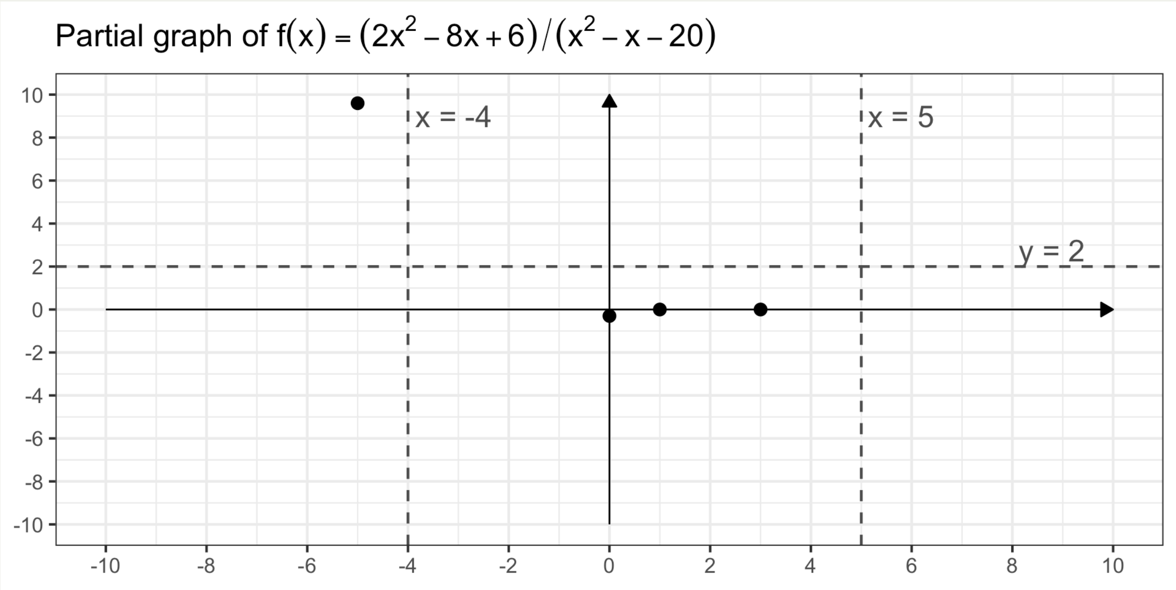
Reference Point at $x = 6$:

$$r(6) = \frac{2(6)^2 - 8(6) + 6}{(6)^2 - (6) - 20} = \frac{30}{10}$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

- vertical asymptotes at $x = -4$ and $x = 5$,
- a horizontal asymptote at $y = 2$,
- roots at $(1, 0)$ and $(3, 0)$, and
- a y-intercept at $(0, -3/10)$



To complete the sketch, we'll need reference points from each of the domain regions. We'll find one to the left of $x = -4$ and one to the right of $x = 5$ because we already have reference points in the middle section.

Reference Point at $x = -5$:

$$r(-5) = \frac{2(-5)^2 - 8(-5) + 6}{(-5)^2 - (-5) - 20} = \frac{48}{5} > 2$$

Reference Point at $x = 6$:

$$r(6) = \frac{2(6)^2 - 8(6) + 6}{(6)^2 - (6) - 20} = 3$$

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

Sketching the Complete Graph

We now know a great deal about $r(x) = \dfrac{2x^2 - 8x + 6}{x^2 - x - 20}$. It has…

Sketching the Graph Practice

Try It! 4: Use your findings from the earlier **Try It!** activities to sketch the graph of $g\left(x\right)=\dfrac{-4x^2-4x+24}{5x^2+5x-100}$. Be sure to include any holes, vertical asymptotes, horizontal asymptote, roots, the y-intercept, and at least one reference point from each region.

Before sketching: Based on your analysis of the function $g\left(x\right)$ so far, how many separate "pieces" will the graph have? In which region will each root appear?

The Full Strategy

Strategy (Analyzing and Graphing a Rational Function): Consider the rational function $r(x) = \dfrac{p(x)}{q(x)}$.

The Full Strategy Practice

Try It! 5: Conduct a full analysis of the rational function $h(x) = \dfrac{x^4 - 7x^3 - 9x^2 + 115x - 100}{x^4 - 7x^3 - 25x^2 + 151x - 120}$

Hints. The expression $x^4 - 7x^3 - 9x^2 + 115x - 100$ factors into $(x+4)(x-1)(x-5)^2$. Similarly, the expression $x^4 - 7x^3 - 25x^2 + 151x - 120$ factors into $(x+5)(x-1)(x-3)(x-8)$.

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `14. Rational Functions and Equations`

Task: Consider $j(x) = \dfrac{x-4}{x^2-x-6}$.

(a) Find the domain of $j(x)$.

(b) Find the roots of $j(x)$, if any exist.

(c) Find any vertical and horizontal asymptotes.

(d) Sketch an approximate graph of $j(x)$.

Summary and Next Time...

Ideas From Today

- A **rational function** $r(x) = \dfrac{p(x)}{q(x)}$ is undefined where $q(x) = 0$ — those are domain restrictions.
- **Vertical asymptotes** occur at $x = a$ where $q(a) = 0$ but $p(a) \neq 0$.
- **Holes** occur at $x = a$, where $q(a) = 0$ and $p(a) = 0$.
- **Horizontal asymptotes** describe end behavior. A rational function may have zero or one horizontal asymptote. Find it by evaluating $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)}$.
- **Roots** come from the numerator: solve $p(x) = 0$.
- Use **reference points** between asymptotes and roots to determine the location of the graph of $r(x)$ in each domain region, then sketch.

Looking Ahead

- We'll continue our exploration of different function classes with **radical functions**.
- This class of function introduces us to a new type of domain restriction.
- Our analysis will include more of the same – domain, y-intercept, roots, end behavior, etc.

Next Time:
Radical Functions
Homework:
Start Homework 10 on MyOpenMath