MAT 142: Polynomial Functions and Equations (Part I)

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Reminders

Objectives

Today we generalize the ideas from *linear* and *quadratic* functions to the broader class of *polynomial functions*.

After today's class meeting, you should be able to:

- Identify the leading term, leading coefficient, and degree of a polynomial.
- Solve polynomial equations that are in (or can be put into) factored form.
- Identify roots of a polynomial function and their multiplicities.
- Describe the behavior of a polynomial graph at each root based on multiplicity.
- Determine the end behavior of a polynomial function using limits.
- Combine all of the above to sketch the graph of a polynomial function.

Polynomial Equations

Definition (Polynomial Equation): A polynomial equation can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

For a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$:

- The **leading term** is $a_n x^n$ – the term with the highest power.
- The **leading coefficient** is a_n – the coefficient on the highest power.
- The **degree** is n – the highest power of x.

Linear ($n = 1$) and quadratic ($n = 2$) equations are both special cases of polynomials.

Solving Polynomial Equations

Strategy: The general approach for solving polynomial equations is the same as for quadratics:

1. Write the equation with 0 on one side.
2. Factor the expression completely.
3. Apply the **zero product property**: if $ab \cdots = 0$, then at least one factor equals 0.

Number of Solutions

A degree n polynomial equation has *at most* n real solutions. Counting complex solutions and multiplicities, it has *exactly* n solutions.

Solving Polynomial Equations I

Example 1: The polynomial equation $4x^3 - 10x^2 - 42x + 40 = 0$ can be partially factored into $(2x+5)\left(2x^2 - 10x + 8\right) = 0$. Find all solutions to the equation.

Solution. We already have 0 on the right side. Continue factoring the left side.

$$(2x+5)\left(2x^2 - 10x + 8\right) = 0$$

Solving Polynomial Equations I

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Solution. We already have 0 on the right side. Continue factoring the left side.

$$\begin{aligned} & (2x+5)\left(2x^2 - 10x + 8\right) = 0 \\ \implies & 2(2x+5)\left(x^2 - 5x + 4\right) = 0 \end{aligned}$$

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Solution. We already have 0 on the right side. Continue factoring the left side.

$$\begin{aligned} (2x+5)\left(2x^2 - 10x + 8\right) &= 0 \\ \implies 2\,(2x+5)\left(x^2 - 5x + 4\right) &= 0 \\ \implies 2\,(2x+5)\,(x-4)\,(x-1) &= 0 \end{aligned}$$

From factored form, we use the zero product property.

$$2x + 5 = 0 \qquad \text{or} \qquad x - 4 = 0 \qquad \text{or} \qquad x - 1 = 0$$

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$$\begin{aligned} (2x+5)\left(2x^2 - 10x + 8\right) &= 0 \\ \implies 2(2x+5)\left(x^2 - 5x + 4\right) &= 0 \\ \implies 2(2x+5)(x-4)(x-1) &= 0 \end{aligned}$$

From factored form, we use the zero product property.

$$\begin{aligned} 2x + 5 &= 0 \qquad \text{or} \qquad & x - 4 &= 0 \qquad \text{or} \qquad & x - 1 &= 0 \\ \implies 2x &= -5 \qquad \text{or} \qquad & x &= 4 \qquad \text{or} \qquad & x &= 1 \end{aligned}$$

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$$\begin{aligned} 2x + 5 &= 0 & &\text{or} & x - 4 &= 0 & &\text{or} & x - 1 &= 0 \\ \implies 2x &= -5 & &\text{or} & x &= 4 & &\text{or} & x &= 1 \\ \implies x &= \frac{-5}{2} \end{aligned}$$

So we have three distinct solutions, $\boxed{x = -5/2}$, $\boxed{x = 4}$, or $\boxed{x = 1}$.

Notice that this was a degree-three polynomial, so three solutions is exactly what we expect (though those solutions may not all be distinct, or real, generally).

Multiplicity of a Root/Solution

Definition (Multiplicity): If $(x - a)^k$ appears in the fully factored form of a polynomial, then $x = a$ is a root of *multiplicity* k.

For example, the equation $x^2 + 6x + 9 = 0$ has factored form $(x + 3)^2 = 0$, and has only one solution. That solution is $x = -3$, but -3 is a *root of multiplicity two*. It results from a factor that appears twice.

Counting multiplicities and complex solutions, a degree n polynomial always has exactly n solutions.

Solutions and their Multiplicities

Example 2: Find all solutions to $\left(x^2-5x+6\right)\left(x^2+x+4\right)(x-8)^3=0$, including complex roots and multiplicities.

Solution.

$$\left(x^2-5x+6\right)\left(x^2+x+4\right)(x-8)^3=0$$

Solutions and their Multiplicities

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Solution.

$$\begin{aligned} \left(x^2-5x+6\right)\left(x^2+x+4\right)(x-8)^3&=0 \\ \implies (x-3)(x-2)\left(x^2+x+4\right)(x-8)^3&=0 \end{aligned}$$

Reading the solutions directly from the *linear* factors, we have $x=3$ (multiplicity 1), $x=2$ (multiplicity 1), $x=8$ (multiplicity 3).

That accounts for $1+1+3=5$ solutions.

The total degree of the polynomial is $2+2+3=7$ (remember that we add exponents when multiplying factors with like bases, so $x^2\cdot x^2\cdot x^3=x^7$).

Because of this, two solutions remain, and we know that they come from $x^2+x+4=0$:

$$x=\frac{-1\pm\sqrt{(1-16)}}{2}$$

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Solution.

$$\begin{aligned} & \left(x^2-5x+6\right)\left(x^2+x+4\right)(x-8)^3=0 \\ \Longrightarrow\ & (x-3)(x-2)\left(x^2+x+4\right)(x-8)^3=0 \end{aligned}$$

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$$x=\frac{-1\pm\sqrt{(1-16)}}{2}=\frac{-1\pm\sqrt{-15}}{2}$$

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Because of this, two solutions remain, and we know that they come from $x^2+x+4=0$:

$$x=\frac{-1\pm\sqrt{(1-16)}}{2}=\frac{-1\pm\sqrt{-15}}{2}=\frac{-1\pm i\sqrt{15}}{2}=\frac{-1}{2}\pm\frac{\sqrt{15}}{2}i$$

The two complex solutions are a conjugate pair, as expected. All 7 solutions accounted for. ✓

Solving Polynomial Equations Practice

Try It! 1: Find all solutions to $\left(x^2+9\right)\left(x^2-11x+30\right)^2=0$.

Before solving, answer: what is the degree of this polynomial? How many solutions should you find (counting multiplicities and complex solutions)?

Reminder: Check that your solution count matches the degree when you're done.

Polynomial Functions

Definition (Polynomial Function): A function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a *polynomial function*. Its graph is a smooth, continuous curve.

To sketch the graph of a polynomial function, we identify five things:

- **y-intercept:** evaluate $f(0)$
- **Roots and multiplicities:** solve $f(x) = 0$
- **Root behavior:** The multiplicity of each root determines the function's behavior at that point.
  - If the root $x = a$ is a root with *odd* multiplicity, the graph of $f(x)$ *passes through* the x-axis at that root.
  - If the root $x = a$ is a root with *even* multiplicity, the graph of $f(x)$ *bounces off* of the x-axis at that root.
- **End behavior:** analyze using limits by either focusing on the leading term or the sign on each of the factors in factored form.
  - $\lim_{x \to -\infty} f(x)$
  - $\lim_{x \to +\infty} f(x)$
- **Rough shape:** connect the dots using the root behavior, y-intercept, and end behavior.

Roots and Multiplicity – Graph Behavior

The multiplicity of a root determines how the graph behaves at that x-intercept:

- If the root $x = a$ is a root with *odd* multiplicity, the graph of $f(x)$ *passes through* the x-axis at that root.
- If the root $x = a$ is a root with *even* multiplicity, the graph of $f(x)$ *bounces off* of the x-axis at that root.

$$f(x) = x^3 (x - 1)(x + 2)^2$$

End Behavior of Polynomial Functions

Just as with quadratics, the **leading term** $a_n x^n$ determines end behavior.

Leading coefficient	**Degree**	$\lim\limits_{x \to -\infty}$	$\lim\limits_{x \to \infty}$
$a_n > 0$	even	$+\infty$	$+\infty$
$a_n < 0$	even	$-\infty$	$-\infty$
$a_n > 0$	odd	$-\infty$	$+\infty$
$a_n < 0$	odd	$+\infty$	$-\infty$

From Factored Form: If your polynomial function is in factored form, for example $f(x) = (x - a_1)^{n_1}(x - a_2)^{n_2} \cdots (x - a_k)^{n_k}$, then analyse the limiting *sign* of each factor when x approaches $-\infty$ and again when x approaches ∞, and determine whether the ultimate product will be positive or negative.

Remember Your "Book Functions"

- Even-degree polynomials behave like parabolas (both ends go the same direction).
- Odd-degree polynomials behave like cubics (ends go opposite directions).

Graphing a Polynomial

Example 3: For $g(x) = (x+5)(x+1)^2(x-3)(x-5)(x-7)^4$, find the y-intercept, roots and their multiplicities, determine the end behavior, and sketch an approximate graph.

Solution.

y-intercept: $g(0) = (0+5)(0+1)^2(0-3)(0-5)(0-7)^4$

Graphing a Polynomial

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Solution.

y-intercept: $g(0) = 5 \cdot (1)^2 \cdot (-3) \cdot (-5) \cdot (-7)^4$

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Solution.

y-intercept: $g(0) = 5 \cdot 1 \cdot (-3) \cdot (-5) \cdot 2401$

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Solution.

y-intercept: $g(0) = 180{,}075$

Roots and multiplicities:

- $x = -5$: multiplicity 1, so passes through
- $x = -1$: multiplicity 2, so bounces
- $x = 3$: multiplicity 1, so passes through
- $x = 5$: multiplicity 1, so passes through
- $x = 7$: multiplicity 4, so bounces

End behavior: We'll evaluate $\lim_{x \to -\infty} g(x)$ and $\lim_{x \to \infty} g(x)$ separately.

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} (x+5)(x+1)^2(x-3)(x-5)(x-7)^4$$

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$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} (x+5)(x+1)^2(x-3)(x-5)(x-7)^4$$

$$= \lim_{x\to-\infty} \overset{(-)}{(x+5)}\overset{(+)}{(x+1)^2}\overset{(-)}{(x-3)}\overset{(-)}{(x-5)}\overset{(+)}{(x-7)^4}$$

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&= -\infty
\end{aligned}$$

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Example 3: For $g(x) = (x+5)(x+1)^2(x-3)(x-5)(x-7)^4$, find the y-intercept, roots and their multiplicities, determine the end behavior, and sketch an approximate graph.

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&= -\infty \\
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\end{aligned}$$

Graphing a Polynomial

Example 3: For $g(x) = (x+5)(x+1)^2(x-3)(x-5)(x-7)^4$, find the y-intercept, roots and their multiplicities, determine the end behavior, and sketch an approximate graph.

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&= \lim_{x\to\infty} (\overset{(+)}{x+5})(\overset{(+)}{x+1})^2(\overset{(+)}{x-3})(\overset{(+)}{x-5})(\overset{(+)}{x-7})^4 \\
&= \infty
\end{aligned}$$

Graphing a Polynomial

Example 3: For $g(x) = (x+5)(x+1)^2(x-3)(x-5)(x-7)^4$, find the y-intercept, roots and their multiplicities, determine the end behavior, and sketch an approximate graph.

Solution.

y-intercept: $g(0) = 180{,}075$

Graphing Polynomials Practice

Try It! 2: For the polynomial function $f(x) = (x+3)^2 (x-1)(x-4)^3$:

- Find the y-intercept.
- List all roots and their multiplicities. Does each root pass through or bounce off the x-axis?
- Determine the end behavior using limits.
- Sketch a rough graph.

Try It! 3: *Before sketching anything* – a sixth degree polynomial function has roots at $x = -2$ (multiplicity 1), $x = 0$ (multiplicity 3), and $x = 4$ (multiplicity 2), with positive leading coefficient.

Without computing anything, predict:

- What happens at each root?
- What are the end behavior limits?
- Does the graph cross the x-axis more times than there are roots, or fewer?

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `12. Polynomial Functions and Equations`

Task: Consider $f(x) = (x+2)^3 (x-1)(x-4)^2$.

(a) What is the degree of f?

(b) List the roots and their multiplicities. At each root, does the graph pass through or bounce off the x-axis?

(c) Determine $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$.

Summary and Next Time…

Ideas From Today

- A **polynomial equation** of degree n has at most n real solutions. Counting complex solutions and multiplicities, it has exactly n.
- **Solving/Root-Finding Strategy:** get 0 on one side, factor completely, apply the zero product property.
- A root of **odd multiplicity** passes through the x-axis; a root of **even multiplicity** bounces off.
- **End behavior** is determined using limits.
    - From expanded form, the leading term dominates: even-degree polynomials have both ends going the same direction; odd-degree polynomials have ends going opposite directions.
    - From factored form, analyse the sign of each term and determine whether the product will be positive or negative.

Looking Ahead

- Today we could only solve polynomial equations that were already in factored (or near-factored) form. What if the polynomial is given in standard, expanded form?
- Next class we'll develop two tools for that situation: the *Rational Roots Theorem* (which narrows down candidates for roots) and *polynomial long division* (which reduces the degree once a root is found).

Next Time:
Polynomial Long Division and Factoring
Homework:
Start Homework 9 on MyOpenMath