

MAT 142: Quadratic Functions and Equations (Part II)

Dr. Gilbert

June 4, 2026

Reminders

At our last meeting, we covered quadratic equations and functions:

- Solving quadratic equations by factoring.
- Solving quadratic equations using the quadratic formula.
- Concavity, vertex, roots, end behavior, and graphing parabolas.

Try the following warm-up problems.

Problem 1: Find the vertex and roots of $f(x) = 3x^2 - 5x - 2$, and use them to sketch a graph. Does the concavity match what you'd expect from the leading coefficient?

Problem 2: Describe how the graphs of each of the functions below can be obtained via transformations of the “book function” $b(x) = x^2$. Then expand each function into standard form $ax^2 + bx + c$. Can you still easily tell the structure of the graph from the standard form?

$$(a) \ g(x) = 2(x - 3)^2 + 7$$

$$(b) \ h(x) = -(x + 5)^2 - 1$$

Note: Problem 2 previews today's main technique. We'll be attempting to obtain this convenient “*vertex form*” from the standard form.

Objectives

Today we develop the third and final method for solving quadratic equations, and connect it to the transformations framework.

After today's class meeting, you should be able to:

- Use *completing the square* to solve a quadratic equation.
- Rewrite a quadratic function in *vertex form* $a(x - h)^2 + k$ by completing the square.
- Identify the vertex (h, k) directly from vertex form and describe the graph as a transformation of $f(x) = x^2$.
- Apply completing the square to find the center and radius of a circle from its general equation.
- Use the distance and midpoint formulas to find the equation of a circle from its diameter endpoints.

Method III: Completing the Square

Key Idea: A *perfect square trinomial* has the form $x^2 + bx + \left(\frac{b}{2}\right)^2$ because it can be factored into $\left(x + \frac{b}{2}\right)^2$.

To solve $ax^2 + bx + c = 0$ by completing the square:

Completing the Square I

Example 1: Use completing the square to solve $x^2 - 8x - 5 = 0$.

Solution. Since $a = 1$, we don't need to factor out a leading coefficient.

$$x^2 - 8x - 5 = 0$$

Ensure 0 is on one side

Completing the Square I

Example 1: Use completing the square to solve $x^2 - 8x - 5 = 0$.

Solution. Since $a = 1$, we don't need to factor out a leading coefficient.

$$\begin{aligned} & x^2 - 8x - 5 = 0 \\ \implies & (x^2 - 8x) - 5 = 0 \end{aligned}$$

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Group variable terms together

Completing the Square I

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Solution. Since $a = 1$, we don't need to factor out a leading coefficient.

$$x^2 - 8x - 5 = 0$$

$$\implies (x^2 - 8x) - 5 = 0$$

$$\begin{array}{l} \left(\frac{-8}{2}\right)^2 = 16 \\ \implies (x^2 - 8x + 16 - 16) - 5 = 0 \end{array}$$

Ensure 0 is on one side

Group variable terms together

Add and subtract $\left(\frac{b}{2}\right)^2$ inside the parentheses

Completing the Square I

Example 1: Use completing the square to solve $x^2 - 8x - 5 = 0$.

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Ensure 0 is on one side

Group variable terms together

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Carefully move the $-\left(\frac{b}{2}\right)^2$ outside the parentheses

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$$\begin{aligned}\left(\frac{-8}{2}\right)^2=16 \\ \implies (x^2 - 8x + 16 - 16) - 5 &= 0 \\ \implies (x^2 - 8x + 16) - 16 - 5 &= 0 \\ \implies (x - 4)^2 - 21 &= 0\end{aligned}$$

Ensure 0 is on one side

Group variable terms together

Add and subtract $\left(\frac{b}{2}\right)^2$ inside the parentheses

Carefully move the $-\left(\frac{b}{2}\right)^2$ outside the parentheses

Factor the trinomial in the parentheses into a perfect

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Group variable terms together

Add and subtract $\left(\frac{b}{2}\right)^2$ inside the parentheses

Carefully move the $-\left(\frac{b}{2}\right)^2$ outside the parentheses

Factor the trinomial in the parentheses into a perfect

Use algebra to isolate the binomial being squared

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$$\begin{aligned}x^2 - 8x - 5 &= 0 \\ \implies (x^2 - 8x) - 5 &= 0 \\ \xRightarrow{\left(\frac{-8}{2}\right)^2=16} (x^2 - 8x + 16 - 16) - 5 &= 0 \\ \implies (x^2 - 8x + 16) - 16 - 5 &= 0 \\ \implies (x - 4)^2 - 21 &= 0 \\ \implies (x - 4)^2 &= 21 \\ \implies x - 4 &= \pm\sqrt{21}\end{aligned}$$

Ensure 0 is on one side

Group variable terms together

Add and subtract $\left(\frac{b}{2}\right)^2$ inside the parentheses

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Use algebra to isolate the binomial being squared

Take the square root of both sides, remembering that

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Solution. Since $a = 1$, we don't need to factor out a leading coefficient.

$$\begin{aligned} & x^2 - 8x - 5 = 0 \\ \implies & (x^2 - 8x) - 5 = 0 \\ \left(\frac{-8}{2}\right)^2 = 16 \implies & (x^2 - 8x + 16 - 16) - 5 = 0 \\ \implies & (x^2 - 8x + 16) - 16 - 5 = 0 \\ \implies & (x - 4)^2 - 21 = 0 \\ \implies & (x - 4)^2 = 21 \\ \implies & x - 4 = \pm\sqrt{21} \\ \implies & x = \boxed{4 \pm \sqrt{21}} \end{aligned}$$

Ensure 0 is on one side

Group variable terms together

Add and subtract $\left(\frac{b}{2}\right)^2$ inside the parentheses

Carefully move the $-\left(\frac{b}{2}\right)^2$ outside the parentheses

Factor the trinomial in the parentheses into a perfect

Use algebra to isolate the binomial being squared

Take the square root of both sides, remembering that

Isolate the variable to find the solutions

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$\begin{aligned} 3x^2 + 5 &= x^2 - 12x \\ \implies 2x^2 + 12x + 5 &= 0 \end{aligned}$$

Get 0 on one side

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

Get 0 on one side

Factor "a" out of the two variable terms

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9 - 9) + 5 = 0$$

Get 0 on one side

Factor "a" out of the two variable terms

Inside parentheses, add and subtract $\left(\frac{6}{2}\right)^2$

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Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9 - 9) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9) - 2(9) + 5 = 0$$

Get 0 on one side

Factor "a" out of the two variable terms

Inside parentheses, add and subtract $\left(\frac{6}{2}\right)^2$

Carefully move the -9 outside, multiplying by the fac

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Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9 - 9) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9) - 2(9) + 5 = 0$$

$$\implies 2(x + 3)^2 - 13 = 0$$

Get 0 on one side

Factor "a" out of the two variable terms

Inside parentheses, add and subtract $\left(\frac{6}{2}\right)^2$

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Factor the grouped terms into a perfect square

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9 - 9) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9) - 2(9) + 5 = 0$$

$$\implies 2(x + 3)^2 - 13 = 0$$

$$\implies (x + 3)^2 = \frac{13}{2}$$

Get 0 on one side

Factor "a" out of the two variable terms

Inside parentheses, add and subtract $\left(\frac{6}{2}\right)^2$

Carefully move the -9 outside, multiplying by the fac

Factor the grouped terms into a perfect square

Isolate the binomial being squared

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9 - 9) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9) - 2(9) + 5 = 0$$

$$\implies 2(x + 3)^2 - 13 = 0$$

$$\implies (x + 3)^2 = \frac{13}{2}$$

$$\implies x + 3 = \pm \sqrt{\frac{13}{2}}$$

Get 0 on one side

Factor "a" out of the two variable terms

Inside parentheses, add and subtract $\left(\frac{6}{2}\right)^2$

Carefully move the -9 outside, multiplying by the fac

Factor the grouped terms into a perfect square

Isolate the binomial being squared

Take the square root, remembering $\pm$

Completing the Square II

Problem: Use completing the square to solve $3x^2 + 5 = x^2 - 12x$.

Solution.

$$3x^2 + 5 = x^2 - 12x$$

$$\implies 2x^2 + 12x + 5 = 0$$

$$\implies 2(x^2 + 6x) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9 - 9) + 5 = 0$$

$$\implies 2(x^2 + 6x + 9) - 2(9) + 5 = 0$$

$$\implies 2(x + 3)^2 - 13 = 0$$

$$\implies (x + 3)^2 = \frac{13}{2}$$

$$\implies x + 3 = \pm \sqrt{\frac{13}{2}}$$

$$\implies x = -3 \pm \sqrt{\frac{13}{2}}$$

Get 0 on one side

Factor "a" out of the two variable terms

Inside parentheses, add and subtract $\left(\frac{6}{2}\right)^2$

Carefully move the -9 outside, multiplying by the fac

Factor the grouped terms into a perfect square

Isolate the binomial being squared

Take the square root, remembering $\pm$

Isolate x to solve

There are two solutions $\boxed{-3 \pm \sqrt{\frac{13}{2}}}$.

Completing the Square Practice

Try It! 1: Use completing the square to solve $x^2 + 6x + 2 = 0$.

Try It! 2: Use completing the square to solve $-3x^2 - 12x + 3 = 0$.

Vertex Form

The same process that solves quadratic equations gives us something even more useful: *vertex form*.

Definition (Vertex Form): A quadratic function written as

$$f(x) = a(x - h)^2 + k$$

is in *vertex form*. The vertex is at (h, k) , and the graph is a transformation of $f(x) = x^2$ which has been

- moved *right* by h units,
- *stretched/compressed* by a factor of a (and *flipped* over the x -axis if $a < 0$), and
- shifted *up* by k units.

This is exactly the transformations framework we encountered earlier this semester.

Now we have a systematic method for getting any quadratic into this form – *complete the square*.

Obtaining Vertex Form

Example 3: Complete the square to rewrite $g(x) = 3x^2 + 36x - 25$ in vertex form. Describe the graph of g as a transformation of $f(x) = x^2$ and identify the vertex.

Solution.

$$g(x) = 3x^2 + 36x - 25$$

Obtaining Vertex Form

Example 3: Complete the square to rewrite $g(x) = 3x^2 + 36x - 25$ in vertex form. Describe the graph of g as a transformation of $f(x) = x^2$ and identify the vertex.

Solution.

$$\begin{aligned} g(x) &= 3x^2 + 36x - 25 \\ &= 3(x^2 + 12x) - 25 \end{aligned}$$

Obtaining Vertex Form

Example 3: Complete the square to rewrite $g(x) = 3x^2 + 36x - 25$ in vertex form. Describe the graph of g as a transformation of $f(x) = x^2$ and identify the vertex.

Solution.

$$\begin{aligned} g(x) &= 3x^2 + 36x - 25 \\ &= 3(x^2 + 12x) - 25 \\ &= 3(x^2 + 12x + 36 - 36) - 25 \end{aligned}$$

Obtaining Vertex Form

Example 3: Complete the square to rewrite $g(x) = 3x^2 + 36x - 25$ in vertex form. Describe the graph of g as a transformation of $f(x) = x^2$ and identify the vertex.

Solution.

$$\begin{aligned} g(x) &= 3x^2 + 36x - 25 \\ &= 3(x^2 + 12x) - 25 \\ &= 3(x^2 + 12x + 36 - 36) - 25 \\ &= 3(x^2 + 12x + 36) - 108 - 25 \end{aligned}$$

Obtaining Vertex Form

Example 3: Complete the square to rewrite $g(x) = 3x^2 + 36x - 25$ in vertex form. Describe the graph of g as a transformation of $f(x) = x^2$ and identify the vertex.

Solution.

$$\begin{aligned} g(x) &= 3x^2 + 36x - 25 \\ &= 3(x^2 + 12x) - 25 \\ &= 3(x^2 + 12x + 36 - 36) - 25 \\ &= 3(x^2 + 12x + 36) - 108 - 25 \\ &= 3(x + 6)^2 - 133 \end{aligned}$$

Obtaining Vertex Form

Example 3: Complete the square to rewrite $g(x) = 3x^2 + 36x - 25$ in vertex form. Describe the graph of g as a transformation of $f(x) = x^2$ and identify the vertex.

Solution.

$$\begin{aligned} g(x) &= 3x^2 + 36x - 25 \\ &= 3(x^2 + 12x) - 25 \\ &= 3(x^2 + 12x + 36 - 36) - 25 \\ &= 3(x^2 + 12x + 36) - 108 - 25 \\ &= \boxed{3(x + 6)^2 - 133} \\ &= \boxed{3(x - (-6))^2 - 133} \end{aligned}$$

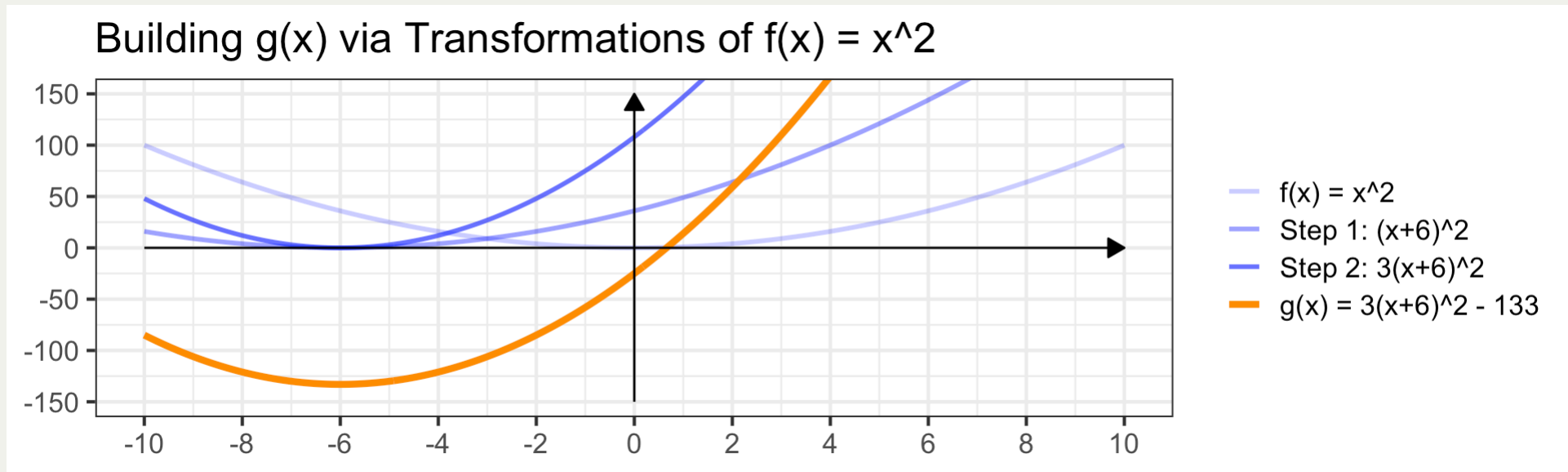
The graph $g(x)$ is moved to the left by **6** and down **133** from the “book function” $f(x) = x^2$. The vertex of the “book function” was at $(0, 0)$, so the vertex of $g(x)$ is at $(-6, -133)$.

Graphing from Vertex Form

Example 4: Describe the graph of $g(x) = 3(x + 6)^2 - 133$ as a transformation of $f(x) = x^2$.

Solution. The transformation sequence applied to the “book function” $f(x) = x^2$:

1. Shift **left** by **6** units: $f(x + 6) = (x + 6)^2$
2. Stretch **vertically** by factor **3** to obtain $3(x + 6)^2$
3. Shift **down** by **133** units and arrive at $g(x) = 3(x + 6)^2 - 133$



Vertex Form Practice

Try It! 3: Complete the square to rewrite $j(x) = 2x^2 - 12x + 7$ in vertex form $a(x - h)^2 + k$.

- Identify the vertex.
- Describe the graph of j as a sequence of transformations applied to $f(x) = x^2$.
- Does the parabola open upward or downward?
- Draw the graph of $j(x)$

Circles

Completing the square has a beautiful application beyond quadratics: finding the **center and radius of a circle** from its general equation.

Definition (Center-Radius Form): The equation of a circle with center (x_0, y_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Example 5: The circle with equation $(x - 4)^2 + (y + 2)^2 = 49$ has *center* at the point $(4, -2)$ and *radius* 7.

When a circle's equation is given in *general form* $(x^2 + bx + y^2 + cy = d)$, we complete the square *twice* — once for the x terms and once for the y terms — to recover the center-radius form.

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$x^2 + 4x + y^2 - 10y = 16$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$\begin{aligned} x^2 + 4x + y^2 - 10y &= 16 \\ \implies (x^2 + 4x) + y^2 - 10y &= 16 \end{aligned}$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$x^2 + 4x + y^2 - 10y = 16$$

$$\implies (x^2 + 4x) + y^2 - 10y = 16$$

$$\implies (x^2 + 4x + 4 - 4) + y^2 - 10y = 16$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

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Complete the Square for x :

$$x^2 + 4x + y^2 - 10y = 16$$

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$$\implies (x^2 + 4x + 4 - 4) + y^2 - 10y = 16$$

$$\implies (x^2 + 4x + 4) - 4 + y^2 - 10y = 16$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$x^2 + 4x + y^2 - 10y = 16$$

$$\implies (x^2 + 4x) + y^2 - 10y = 16$$

$$\implies (x^2 + 4x + 4 - 4) + y^2 - 10y = 16$$

$$\implies (x^2 + 4x + 4) - 4 + y^2 - 10y = 16$$

$$\implies (x + 2)^2 - 4 + y^2 - 10y = 16$$

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Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

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Complete the Square for x :

$$x^2 + 4x + y^2 - 10y = 16$$

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$$\implies (x^2 + 4x + 4 - 4) + y^2 - 10y = 16$$

$$\implies (x^2 + 4x + 4) - 4 + y^2 - 10y = 16$$

$$\implies (x + 2)^2 - 4 + y^2 - 10y = 16$$

$$\implies (x + 2)^2 + y^2 - 10y = 20$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$\begin{aligned} & x^2 + 4x + y^2 - 10y = 16 \\ \implies & (x^2 + 4x) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4 - 4) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4) - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 + y^2 - 10y = 20 \end{aligned}$$

Complete the Square for y :

$$(x + 2)^2 + y^2 - 10y = 20$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$\begin{aligned} & x^2 + 4x + y^2 - 10y = 16 \\ \implies & (x^2 + 4x) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4 - 4) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4) - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 + y^2 - 10y = 20 \end{aligned}$$

Complete the Square for y :

$$\begin{aligned} & (x + 2)^2 + y^2 - 10y = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y) = 20 \end{aligned}$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$\begin{aligned} & x^2 + 4x + y^2 - 10y = 16 \\ \implies & (x^2 + 4x) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4 - 4) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4) - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 + y^2 - 10y = 20 \end{aligned}$$

Complete the Square for y :

$$\begin{aligned} & (x + 2)^2 + y^2 - 10y = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y) = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y + 25 - 25) = 20 \end{aligned}$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

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Complete the Square for y :

$$\begin{aligned} & (x + 2)^2 + y^2 - 10y = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y) = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y + 25 - 25) = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y + 25) - 25 = 20 \end{aligned}$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

$$\begin{aligned} & x^2 + 4x + y^2 - 10y = 16 \\ \implies & (x^2 + 4x) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4 - 4) + y^2 - 10y = 16 \\ \implies & (x^2 + 4x + 4) - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 - 4 + y^2 - 10y = 16 \\ \implies & (x + 2)^2 + y^2 - 10y = 20 \end{aligned}$$

Complete the Square for y :

$$\begin{aligned} & (x + 2)^2 + y^2 - 10y = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y) = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y + 25 - 25) = 20 \\ \implies & (x + 2)^2 + (y^2 - 10y + 25) - 25 = 20 \\ \implies & (x + 2)^2 + (y - 5)^2 - 25 = 20 \end{aligned}$$

Finding the Center and Radius

Example 6: Find the center and radius of the circle given by $x^2 + 4x + y^2 - 10y = 16$.

Solution. We'll complete the square for x and then complete the square for y .

Complete the Square for x :

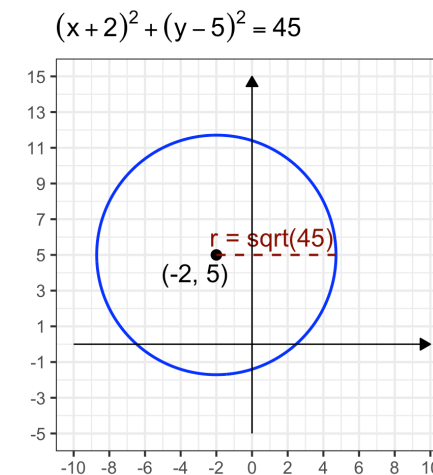
$$\begin{aligned}x^2 + 4x + y^2 - 10y &= 16 \\ \implies (x^2 + 4x) + y^2 - 10y &= 16 \\ \implies (x^2 + 4x + 4 - 4) + y^2 - 10y &= 16 \\ \implies (x^2 + 4x + 4) - 4 + y^2 - 10y &= 16 \\ \implies (x + 2)^2 - 4 + y^2 - 10y &= 16 \\ \implies (x + 2)^2 + y^2 - 10y &= 20\end{aligned}$$

The equation of the circle in its center-radius form is $(x - (-2))^2 + (y - 5)^2 = 45$.

The center of the circle is at the point $(-2, 5)$ and the radius is $\sqrt{45}$.

Complete the Square for y :

$$\begin{aligned}(x + 2)^2 + y^2 - 10y &= 20 \\ \implies (x + 2)^2 + (y^2 - 10y) &= 20 \\ \implies (x + 2)^2 + (y^2 - 10y + 25 - 25) &= 20 \\ \implies (x + 2)^2 + (y^2 - 10y + 25) - 25 &= 20 \\ \implies (x + 2)^2 + (y - 5)^2 - 25 &= 20 \\ \implies (x + 2)^2 + (y - 5)^2 &= 45\end{aligned}$$



Circles Practice

Try It! 4: Find the center and radius of the circle given by $x^2 - 6x + y^2 + 8y = 11$.

Reminder: Complete the square separately for the x terms and the y terms.

Distance and Midpoint Formulas

We have two more tools that come in handy when working with circles and geometry more broadly.

Definition (Distance Formula): The distance between points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Definition (Midpoint Formula): The midpoint between (x_1, y_1) and (x_2, y_2) is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

The distance formula comes from the Pythagorean theorem and the midpoint formula uses the idea of coordinatewise averaging.

Circle From Two Points

Example 7: Find the equation of the circle with $(-3, 1)$ and $(5, 9)$ at opposite ends of a diameter.

Solution. The center is the midpoint of the diameter.

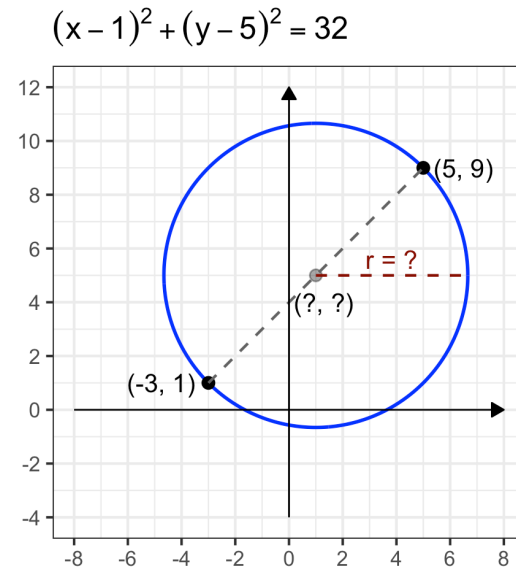
$$\text{Center} = \left(\frac{-3 + 5}{2}, \frac{1 + 9}{2} \right) = (1, 5)$$

The radius is the distance from the center to either endpoint.

$$r = \sqrt{((5 - 1)^2 + (9 - 5)^2)} = \sqrt{32}$$

The equation of the circle is

$$\boxed{(x - 1)^2 + (y - 5)^2 = 32}.$$



Circles Practice

Try It! 5: Find the equation of the circle with $(2, -1)$ and $(8, 7)$ at opposite ends of a diameter.

Reminder: Use the midpoint formula for the center, then the distance formula for the radius.

Exit Ticket Task

Navigate to [our MAT142 Exit Ticket Form](#), answer the questions, and complete the *task* below.

Note. Today's discussion is listed as [11. Quadratic Functions and Equations \(Part II\)](#)



Task: Complete the square to rewrite $f(x) = x^2 - 6x + 2$ in vertex form. State the vertex and describe the graph as a transformation of $f(x) = x^2$.

Summary and Next Time...

Ideas From Today

- **Completing the square** converts $ax^2 + bx + c$ into $a(x - h)^2 + k$ by adding and subtracting $\left(\frac{b}{2a}\right)^2$ and factoring.
- The resulting form $f(x) = a(x - h)^2 + k$ is called **vertex form**
 - The vertex is (h, k) and the graph is a direct transformation of $g(x) = x^2$.
- For circles, completing the square twice recovers **center-radius form** $(x - x_0)^2 + (y - y_0)^2 = r^2$.
- The **distance formula** gives the length between two points; the **midpoint formula** gives the point halfway between them.
- Given the endpoints of a diameter of a circle, use midpoint for the center and distance for the radius.

Looking Ahead

- With quadratics complete, we move into **polynomial functions**. These are a natural generalization beyond *linear* and *quadratic* functions. You'll see that many of the ideas from quadratics (roots, end behavior, leading coefficient, end behavior, etc.) carry forward in a richer form.

Next Time:

Polynomial Functions

Homework:

Complete Homework 8 on MyOpenMath