MAT 142: Quadratic Functions and Equations (Part I)

Dr. Gilbert

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Reminders

At our last meeting, we introduced complex numbers and their arithmetic. We'll see today why complex numbers arise naturally when studying quadratic equations.

Try the following warm-up problems.

Problem 1: Expand and simplify each of the following.

(a) $(x-7)(x-2)$

(b) $(3x-2)(2x+7)$

(c) $(x+4)(x-3)$

(d) $(x-5)^2$

Problem 2: Consider the function $f(x) = -2x^2 - 2x + 24$. Evaluate $f\left(-\frac{1}{2}\right)$, $f(-4)$, and $f(3)$.

Motivating Application

A rocket is launched into the air. Its height (in meters above sea level) as a function of time (in seconds) is given by

$$h\left(t\right) = -4.9t^2 + 229t + 234$$

Find the **maximum height** the rocket attains.

Before we work through this, consider the following:

- What is the domain of $h\left(t\right)$ without considering the real-life context?
- What is the practical domain of $h\left(t\right)$ given the context of the problem?
- Why wouldn't a *linear* model make sense here? What would a linear model assume about the rocket's flight?

As usual, we'll return to this application at the end of today's class.

Objectives

Today's class has two connected halves.

First half – Solving Quadratic Equations:

- Solve quadratic equations via factoring.
- Solve quadratic equations using the quadratic formula.
- Identify when solutions are complex (non-real).

Second half – Quadratic Functions and Their Graphs:

- Identify the leading coefficient and use it to determine concavity.
- Find the vertex of a parabola.
- Find roots using equation-solving techniques.
- Determine end behavior using limits.

Quadratic Equations

Definition (Quadratic Equation): A *quadratic equation* is an equation involving a single variable of degree $\mathbf{2}$. The general form is

$$ax^2 + bx + c = 0$$

Examples of quadratic equations:

- $2x^2 - 4x - 10 = 0$
- $x^2 = 8 - 2x$
- $-3x^2 - x + 5 = 32 - 2x^2$
- $\dfrac{4x^2 + 8x}{5} + 2 = x + 2$
- $(x - 3)(5 - 2x) = x + 8$
- $x^2 = 9$

A quadratic equation may have $\mathbf{0}$, $\mathbf{1}$, or $\mathbf{2}$ solutions.

Method I: Factoring

Strategy (Solving by Factoring):

1. Use algebra to get 0 on one side of the equation.
2. Factor the expression on the other side.
3. Use the *zero product property*, which guarantees that if $ab = 0$, then $a = 0$ or $b = 0$.

When to Factor

Factoring is quick when integer or rational solutions exist, but it won't always be clear whether factors exist. If you can't factor within about a minute, move on to the quadratic formula (which we'll cover next).

Example 1: Solve $x^2 - 9x + 14 = 0$ by factoring.

Solution. Look for two numbers that multiply to 14 and add to -9.

$$x^2 - 9x + 14 = 0$$

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Solution. Look for two numbers that multiply to 14 and add to -9.

$$\begin{aligned} x^2 - 9x + 14 &= 0 \\ \implies (x \mathbin{?} \_\_\_)\,(x \mathbin{?} \_\_\_) &= 0 \end{aligned}$$

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$$\begin{aligned} x^2 - 9x + 14 &= 0 \\ \implies (x \mathbin{?} \underline{\quad}) (x \mathbin{?} \underline{\quad}) &= 0 \\ \implies (x - \underline{\quad}) (x - \underline{\quad}) &= 0 \end{aligned}$$

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Solution. Look for two numbers that multiply to 14 and add to -9.

$$\begin{aligned} x^2 - 9x + 14 &= 0 \\ \implies (x \;?\; \_\_\_)\,(x \;?\; \_\_\_) &= 0 \\ \implies (x - \_\_\_)\,(x - \_\_\_) &= 0 \\ \implies (x-7)\,(x-2) &= 0 \end{aligned}$$

Now, using the *zero product property*, we have $x - 7 = 0$ or $x - 2 = 0$.

$$x - 7 = 0 \qquad \text{or} \qquad x - 2 = 0$$

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Factoring is quick when integer or rational solutions exist, but it won't always be clear whether factors exist. If you can't factor within about a minute, move on to the quadratic formula (which we'll cover next).

Example 1: Solve $x^2 - 9x + 14 = 0$ by factoring.

Solution. Look for two numbers that multiply to 14 and add to -9.

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\begin{aligned}
x^2 - 9x + 14 &= 0 \\
\implies (x \mathbin{?} \underline{\quad})(x \mathbin{?} \underline{\quad}) &= 0 \\
\implies (x - \underline{\quad})(x - \underline{\quad}) &= 0 \\
\implies (x - 7)(x - 2) &= 0
\end{aligned}
$$

Now, using the *zero product property*, we have $x - 7 = 0$ or $x - 2 = 0$.

$$
\begin{aligned}
x - 7 &= 0 \qquad \text{or} \qquad & x - 2 &= 0 \\
\implies x &= 7 \qquad \text{or} \qquad & x &= 2
\end{aligned}
$$

There are two solutions to the quadratic equation. We have $\boxed{x = 7}$ or $\boxed{x = 2}$.

Solving Quadratics by Factoring

Example 2: Solve $4x^2 + 18x - 14 = -2x^2 + x$ by factoring.

Solution. First, collect everything on one side so that 0 appears on the other.

$$4x^2 + 18x - 14 = -2x^2 + x$$

Solving Quadratics by Factoring

Example 2: Solve $4x^2 + 18x - 14 = -2x^2 + x$ by factoring.

Solution. First, collect everything on one side so that 0 appears on the other.

$$\begin{aligned} & 4x^2 + 18x - 14 = -2x^2 + x \\ \implies & 6x^2 + 17x - 14 = 0 \end{aligned}$$

Solving Quadratics by Factoring

Example 2: Solve $4x^2 + 18x - 14 = -2x^2 + x$ by factoring.

Solution. First, collect everything on one side so that 0 appears on the other.

$$\begin{aligned} 4x^2 + 18x - 14 &= -2x^2 + x \\ \implies 6x^2 + 17x - 14 &= 0 \\ \implies (\_\_\_x \; ? \; \_\_\_)(\_\_\_x \; ? \; \_\_\_) &= 0 \end{aligned}$$

Solving Quadratics by Factoring

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Now we'll use the *zero product property* to obtain two linear equations.

$$2x + 7 = 0 \qquad \text{or} \qquad 3x - 2 = 0$$

Solving Quadratics by Factoring

Example 2: Solve $4x^2 + 18x - 14 = -2x^2 + x$ by factoring.

Solution. First, collect everything on one side so that 0 appears on the other.

$$\begin{aligned}
4x^2 + 18x - 14 &= -2x^2 + x \\
\implies 6x^2 + 17x - 14 &= 0 \\
\implies (___x \; ? \; ___)\,(___x \; ? \; ___) &= 0 \\
\implies (___x + ___)\,(___x - ___) &= 0 \\
\implies (2x+7)\,(3x-2) &= 0
\end{aligned}$$

Now we'll use the *zero product property* to obtain two linear equations.

$$\begin{aligned}
2x + 7 &= 0 \qquad \text{or} \qquad & 3x - 2 &= 0 \\
\implies 2x &= -7 \qquad \text{or} \qquad & 3x &= 2
\end{aligned}$$

Solving Quadratics by Factoring

Example 2: Solve $4x^2 + 18x - 14 = -2x^2 + x$ by factoring.

Solution. First, collect everything on one side so that 0 appears on the other.

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Now we'll use the *zero product property* to obtain two linear equations.

$$\begin{aligned} 2x + 7 &= 0 & \text{or} \qquad 3x - 2 &= 0 \\ \implies 2x &= -7 & \text{or} \qquad 3x &= 2 \\ \implies x &= \frac{-7}{2} & \text{or} \qquad x &= \frac{2}{3} \end{aligned}$$

Again, we've found two possible solutions to the quadratic equation. We have either $\boxed{x = \frac{-7}{2}}$ or $\boxed{x = \frac{2}{3}}$.

Factoring Practice

Try It! 1: Solve the quadratic $x^2 + x - 12 = 0$ by factoring.

Try It! 2: Solve the quadratic $2x^2 - x - 6 = 0$ by factoring.

Try It! 3: Solve the quadratic $x^2 - 10x + 25 = 0$ by factoring.

Method II: The Quadratic Formula

Definition (Quadratic Formula): Any solutions to the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

The expression $b^2 - 4ac$ under the square root is called the *discriminant*. It tells us how many real solutions exist:

- If $b^2 - 4ac > 0$, then there are *two distinct real solutions*
- If $b^2 - 4ac = 0$, then there is *exactly one real solution*
- If $b^2 - 4ac < 0$, then there are *no real solutions* (and there will be two complex solutions)

Strategy: Get the equation into the form $ax^2 + bx + c = 0$, identify a, b, c, then substitute into the formula.

Solving a Quadratic with the Quadratic Formula

Example 4: Revisit Example 2. Solve $6x^2 + 17x - 14 = 0$ using the quadratic formula.

Solution. Here $a = 6$, $b = 17$, $c = -14$.

$$x = \frac{-17 \pm \sqrt{\left(17^2 - 4\left(6\right)\left(-14\right)\right)}}{2\left(6\right)}$$

Solving a Quadratic with the Quadratic Formula

Example 4: Revisit Example 2. Solve $6x^2 + 17x - 14 = 0$ using the quadratic formula.

Solution. Here $a = 6$, $b = 17$, $c = -14$.

$$
\begin{aligned}
x &= \frac{-17 \pm \sqrt{\left(17^2 - 4\,(6)\,(-14)\right)}}{2\,(6)} \\
\implies x &= \frac{-17 \pm \sqrt{(289 - (-336))}}{12}
\end{aligned}
$$

Solving a Quadratic with the Quadratic Formula

Example 4: Revisit Example 2. Solve $6x^2 + 17x - 14 = 0$ using the quadratic formula.

Solution. Here $a = 6$, $b = 17$, $c = -14$.

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\begin{aligned}
x &= \frac{-17 \pm \sqrt{\left(17^2 - 4\left(6\right)\left(-14\right)\right)}}{2\left(6\right)} \\
\implies x &= \frac{-17 \pm \sqrt{\left(289 - \left(-336\right)\right)}}{12} \\
\implies x &= \frac{-17 \pm \sqrt{\left(289 + 336\right)}}{12}
\end{aligned}
$$

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x &= \frac{-17 \pm \sqrt{\left(17^2 - 4\,(6)\,(-14)\right)}}{2\,(6)} \\
\implies x &= \frac{-17 \pm \sqrt{(289 - (-336))}}{12} \\
\implies x &= \frac{-17 \pm \sqrt{(289 + 336)}}{12} \\
\implies x &= \frac{-17 \pm \sqrt{625}}{12}
\end{aligned}
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\implies x &= \frac{-17 \pm \sqrt{(289 + 336)}}{12} \\
\implies x &= \frac{-17 \pm \sqrt{625}}{12} \\
\implies x &= \frac{-17 \pm 25}{12}
\end{aligned}
$$

Now, we'll evaluate separately – once for the sum in the numerator and then for the difference.

$$x = \frac{-17 - 25}{12} \qquad \text{or} \qquad x = \frac{-17 + 25}{12}$$

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$$
\begin{aligned}
x &= \frac{-17 - 25}{12} \quad \text{or} \quad x = \frac{-17 + 25}{12} \\
\implies x &= \frac{-42}{12} \quad \text{or} \quad x = \frac{8}{12}
\end{aligned}
$$

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\implies x &= \frac{-42}{12} & \text{or} \qquad x &= \frac{8}{12} \\
\implies x &= \frac{-7}{2} & \text{or} \qquad x &= \frac{2}{3}
\end{aligned}
$$

We've landed in exactly the same spot. There are two possible solutions to the quadratic. Either $\boxed{x = \frac{-7}{2}}$ or $\boxed{x = \frac{2}{3}}$.

Solving a Quadratic with the Quadratic Formula

Example 5: Use the quadratic formula to solve $3x^2 + 5x + 10 = 2x^2 + 3$.

Solution. First, collect everything on one side.

$$3x^2 + 5x + 10 = 2x^2 + 3$$

Solving a Quadratic with the Quadratic Formula

Example 5: Use the quadratic formula to solve $3x^2 + 5x + 10 = 2x^2 + 3$.

Solution. First, collect everything on one side.

$$\begin{aligned} 3x^2 + 5x + 10 &= 2x^2 + 3 \\ \implies x^2 + 5x + 7 &= 0 \end{aligned}$$

Here $a = 1$, $b = 5$, $c = 7$.

$$x = \frac{-5 \pm \sqrt{\left(5^2 - 4\left(1\right)\left(7\right)\right)}}{2\left(1\right)}$$

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Solution. First, collect everything on one side.

$$\begin{aligned} 3x^2 + 5x + 10 &= 2x^2 + 3 \\ \implies x^2 + 5x + 7 &= 0 \end{aligned}$$

Here $a = 1$, $b = 5$, $c = 7$.

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{\left(5^2 - 4\left(1\right)\left(7\right)\right)}}{2\left(1\right)} \\ \implies x &= \frac{-5 \pm \sqrt{\left(25 - 28\right)}}{2} \end{aligned}$$

Solving a Quadratic with the Quadratic Formula

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$$\begin{aligned} x &= \frac{-5 \pm \sqrt{\left(5^2 - 4\left(1\right)\left(7\right)\right)}}{2\left(1\right)} \\ \implies x &= \frac{-5 \pm \sqrt{\left(25 - 28\right)}}{2} \\ \implies x &= \frac{-5 \pm \sqrt{-3}}{2} \end{aligned}$$

The discriminant is negative, so there are **no real solutions**.

While there are no real solutions, there will be complex solutions instead.

We'll show them explicitly here.

$$x = \frac{-5 - \sqrt{-3}}{2} \qquad \text{or} \qquad x = \frac{-5 + \sqrt{-3}}{2}$$

Solving a Quadratic with the Quadratic Formula

Example 5: Use the quadratic formula to solve $3x^2 + 5x + 10 = 2x^2 + 3$.

Solution. First, collect everything on one side.

$$\begin{aligned} 3x^2 + 5x + 10 &= 2x^2 + 3 \\ \Longrightarrow \quad x^2 + 5x + 7 &= 0 \end{aligned}$$

Here $a = 1$, $b = 5$, $c = 7$.

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{\left(5^2 - 4\,(1)\,(7)\right)}}{2\,(1)} \\ \Longrightarrow \quad x &= \frac{-5 \pm \sqrt{(25 - 28)}}{2} \\ \Longrightarrow \quad x &= \frac{-5 \pm \sqrt{-3}}{2} \end{aligned}$$

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$$\begin{aligned} x &= \frac{-5 - \sqrt{-3}}{2} \qquad \text{or} \qquad x = \frac{-5 + \sqrt{}}{2} \\ \Longrightarrow \quad x &= \frac{-5 - \sqrt{((3)\,(-1))}}{2} \qquad \text{or} \qquad x = \frac{-5 + \sqrt{}}{} \end{aligned}$$

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Example 5: Use the quadratic formula to solve $3x^2 + 5x + 10 = 2x^2 + 3$.

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\implies \; & x^2 + 5x + 7 = 0
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\implies \; & x = \frac{-5 \pm \sqrt{(25 - 28)}}{2} \\
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\begin{aligned}
& x = \frac{-5 - \sqrt{-3}}{2} \quad \text{or} \quad x = \frac{-5 + \sqrt{}}{2} \\
\implies \; & x = \frac{-5 - \sqrt{((3)\,(-1))}}{2} \quad \text{or} \quad x = \frac{-5 + \sqrt{}}{} \\
\implies \; & x = \frac{-5 - i\sqrt{3}}{2} \quad \text{or} \quad x = \frac{-5 + i\sqrt{}}{2}
\end{aligned}
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Solving a Quadratic with the Quadratic Formula

Example 5: Use the quadratic formula to solve $3x^2 + 5x + 10 = 2x^2 + 3$.

Solution. First, collect everything on one side.

$$\begin{aligned} 3x^2 + 5x + 10 &= 2x^2 + 3 \\ \implies x^2 + 5x + 7 &= 0 \end{aligned}$$

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$$\begin{aligned} x &= \frac{-5 \pm \sqrt{\left(5^2 - 4\left(1\right)\left(7\right)\right)}}{2\left(1\right)} \\ \implies x &= \frac{-5 \pm \sqrt{\left(25 - 28\right)}}{2} \\ \implies x &= \frac{-5 \pm \sqrt{-3}}{2} \end{aligned}$$

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While there are no real solutions, there will be complex solutions instead.

We'll show them explicitly here.

$$\begin{aligned} x &= \frac{-5 - \sqrt{-3}}{2} \quad \text{or} \quad x = \frac{-5 + \sqrt{}}{2} \\ \implies x &= \frac{-5 - \sqrt{\left(\left(3\right)\left(-1\right)\right)}}{2} \quad \text{or} \quad x = \frac{-5 + \sqrt{}}{} \\ \implies x &= \frac{-5 - i\sqrt{3}}{2} \quad \text{or} \quad x = \frac{-5 + i\sqrt{}}{2} \\ \implies x &= \frac{-5}{2} - \frac{\sqrt{3}}{2}i \quad \text{or} \quad x = \frac{-5}{2} + \end{aligned}$$

The two solutions are *complex conjugates*, which will always be the case.

Quadratic Formula Practice

Try It! 4: Solve $x^2 - 4x - 5 = 0$ using the quadratic formula.

Try It! 5: Solve $2x^2 + 2x + 5 = 1 - x$ using the quadratic formula.

Try It! 6: Solve $x^2 = 6x - 9$ using the quadratic formula.

Quadratic Functions and Their Graphs

Definition (Quadratic Function): A *quadratic function* can be written in the form

$$f(x) = ax^2 + bx + c$$

Its graph is a *parabola*. This is the "U"-shaped "book function" $f(x) = x^2$ that you already know, but possibly transformed.

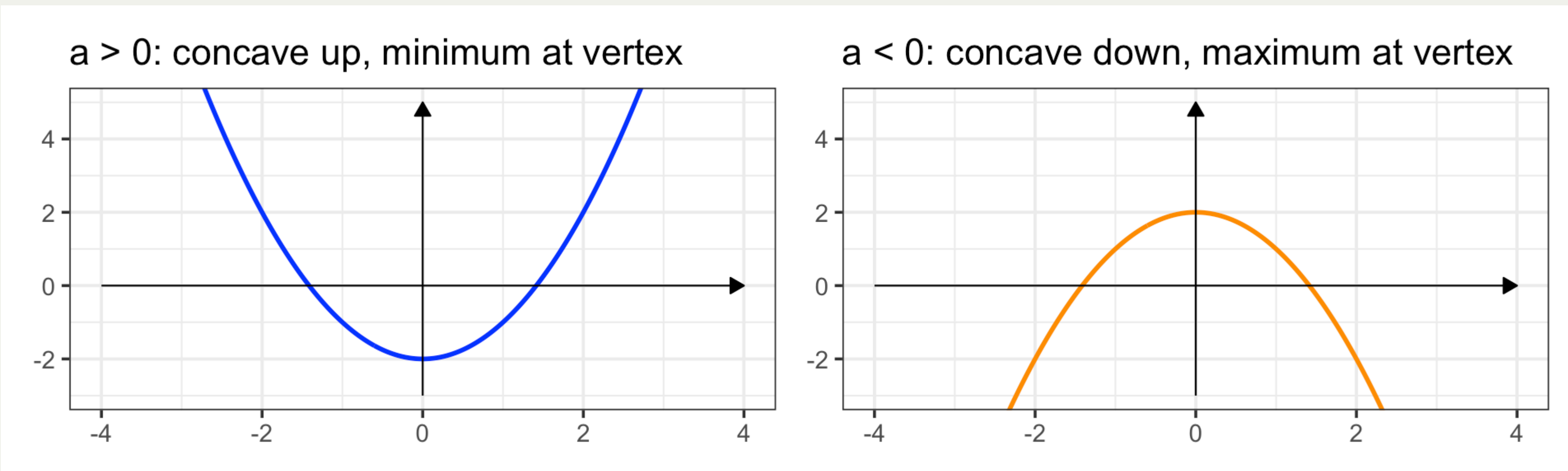
The key features of a parabola:

- **Concavity:** describes the direction (up/down) that the parabola opens and is determined by the sign of the leading coefficient, a
- **Vertex:** the turn-around point on the parabola (its maximum or minimum)
- **Roots:** the locations where $f(x) = 0$ (there may be 0, 1, or 2)
- **End behavior:** what happens as $x \to \pm\infty$

Concavity and the Leading Coefficient

Definition (Leading Coefficient and Concavity): For $f(x) = ax^2 + bx + c$, the *leading coefficient* is a.

- If $a > 0$: the parabola opens *upward* (*concave up*) and the vertex is a *minimum*.
- If $a < 0$: the parabola opens *downward* (*concave down*) and the vertex is a *maximum*.



Example 6: Determine the concavity of the quadratic function $f(x) = -2x^2 - 8x + 4$.

Solution. Since the *leading coefficient* (the coefficient on x^2) is negative (-2), the function is concave *down*.

The Vertex

Definition (Vertex): The vertex of the parabola $f(x) = ax^2 + bx + c$ is located at

$$\left(\frac{-b}{2a},\ f\left(\frac{-b}{2a}\right)\right)$$

The x-coordinate $\dfrac{-b}{2a}$ is the axis of symmetry. The y-coordinate is found by evaluating f at that x-value.

Example 7: Determine the vertex and the *axis of symmetry* of the quadratic function $f(x) = -2x^2 - 8x + 4$.

Solution. The x-coordinate of the vertex is at
$x = \dfrac{-b}{2a}$.

$$x = \frac{8}{2(-2)}$$

The Vertex

Definition (Vertex): The vertex of the parabola $f(x) = ax^2 + bx + c$ is located at

$$\left(\frac{-b}{2a},\ f\left(\frac{-b}{2a}\right)\right)$$

The x-coordinate $\dfrac{-b}{2a}$ is the axis of symmetry. The y-coordinate is found by evaluating f at that x-value.

Example 7: Determine the vertex and the *axis of symmetry* of the quadratic function $f(x) = -2x^2 - 8x + 4$.

Solution. The x-coordinate of the vertex is at $x = \dfrac{-b}{2a}$.

$$\begin{aligned} x &= \frac{8}{2(-2)} \\ \implies x &= \frac{8}{-4} \end{aligned}$$

The Vertex

Definition (Vertex): The vertex of the parabola $f(x) = ax^2 + bx + c$ is located at

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a} \right) \right)$$

The x-coordinate $\dfrac{-b}{2a}$ is the axis of symmetry. The y-coordinate is found by evaluating f at that x-value.

Example 7: Determine the vertex and the *axis of symmetry* of the quadratic function $f(x) = -2x^2 - 8x + 4$.

Solution. The x-coordinate of the vertex is at $x = \dfrac{-b}{2a}$.

$$\begin{aligned} x &= \frac{8}{2(-2)} \\ \implies x &= \frac{8}{-4} \\ \implies x &= -2 \end{aligned}$$

So the x-coordinate of the vertex is at $x = -2$, and we find the y-coordinate of the vertex by evaluating $f(-2)$.

$$f(-2) = -2(-2)^2 - 8(-2) + 4$$

The Vertex

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$$\begin{aligned} f(-2) &= -2(-2)^2 - 8(-2) + 4 \\ &= -8 + 16 + 4 \end{aligned}$$

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$$\begin{aligned} f(-2) &= -2(-2)^2 - 8(-2) + 4 \\ &= -8 + 16 + 4 \\ &= 12 \end{aligned}$$

The vertex of the parabola is at the point $(-2, 12)$ and the axis of symmetry is at $x = -2$.

End Behavior

Definition (End Behavior): The end behavior of $f(x)$ describes what happens as $x \to \infty$ and $x \to -\infty$, expressed using limits:

$$\lim_{x \to -\infty} f(x) \qquad \text{and} \qquad \lim_{x \to \infty} f(x)$$

For quadratic functions $f(x) = ax^2 + bx + c$, the **leading term** ax^2 dominates as x becomes large in magnitude. So:

- If $a > 0$: $\lim\limits_{x \to -\infty} ax^2 + bx + c = \infty$ and $\lim\limits_{x \to \infty} ax^2 + bx + c = \infty$
- If $a < 0$: $\lim\limits_{x \to -\infty} ax^2 + bx + c = -\infty$ and $\lim\limits_{x \to \infty} ax^2 + bx + c = -\infty$

Example 8: Determine the end behavior of the quadratic function $f(x) = -2x^2 - 8x + 4$.

Solution.

- $\lim\limits_{x \to -\infty} -2x^2 - 8x + 4 = -\infty$
- $\lim\limits_{x \to \infty} -2x^2 - 8x + 4 = -\infty$

Note that this behavior aligns with all of the characteristics we've discovered so far.

- The parabola is *concave down* (it opens downwards)
- The vertex is a *maximum*
- The only possibility for the end behavior of this "U"-shaped function was to extend to negative infinity.

Finding Roots and y-Intercept

We know a lot about the function $f(x) = -2x^2 - 8x + 4$ from the previous few examples.

The last thing we need before we can graph the function is to determine its roots.

Example 9: Find the roots and y-intercept of the function $f(x) = -2x^2 - 8x + 4$.

Solution. We'll solve $f(x) = 0$, just like we did earlier today.

$$-2x^2 - 8x + 4 = 0$$

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Solution. We'll solve $f(x) = 0$, just like we did earlier today.

$$\begin{aligned} -2x^2 - 8x + 4 &= 0 \\ \implies -2\left(x^2 + 4x - 2\right) &= 0 \end{aligned}$$

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After deciding against factoring, we'll use the quadratic formula.

$$x = \frac{-4 \pm \sqrt{\left(4^2 - 4(1)(-2)\right)}}{2(1)}$$

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$$\implies x = \frac{-4 \pm \sqrt{(16 + 8)}}{2}$$

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These roots, $\boxed{x = -2 \pm \sqrt{6}}$ are real but won't simplify nicely. We'll just leave it here!

The y-intercept is much simpler to find. We'll just evaluate $f(0)$.

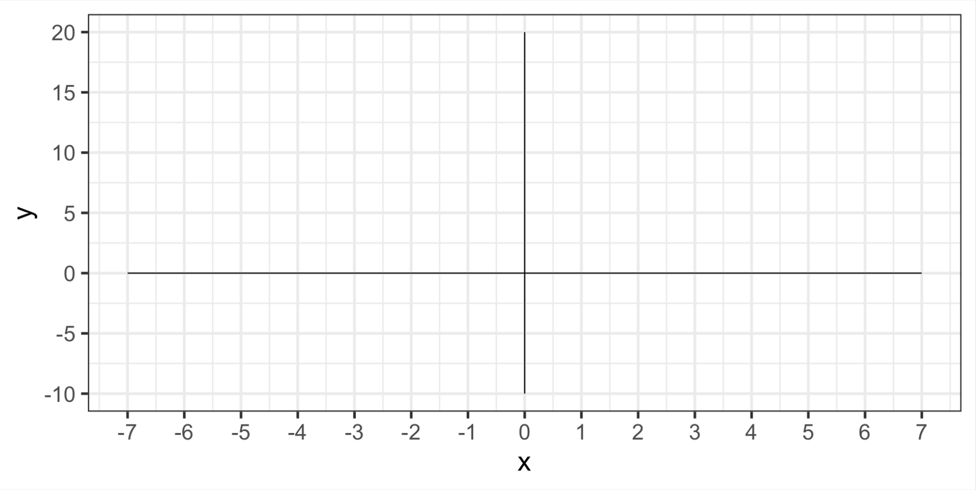
Since $f(0) = 4$, the y-intercept is at the point $\boxed{(0, 4)}$.

Putting It All Together

Example 10: Use information about the concavity, roots, y-intercept, vertex, axis of symmetry, and end behavior of the function $f(x) = -2x^2 - 8x + 4$ to draw its graph.

Solution.

Over the last several examples, we've investigated all of these items.



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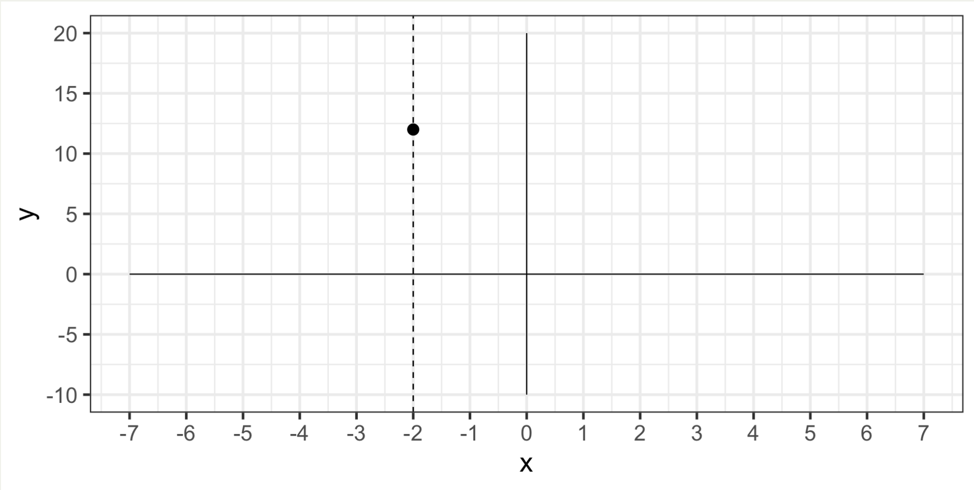
- The vertex is at the point $(-2, 12)$ and the axis of symmetry is at $x = -2$



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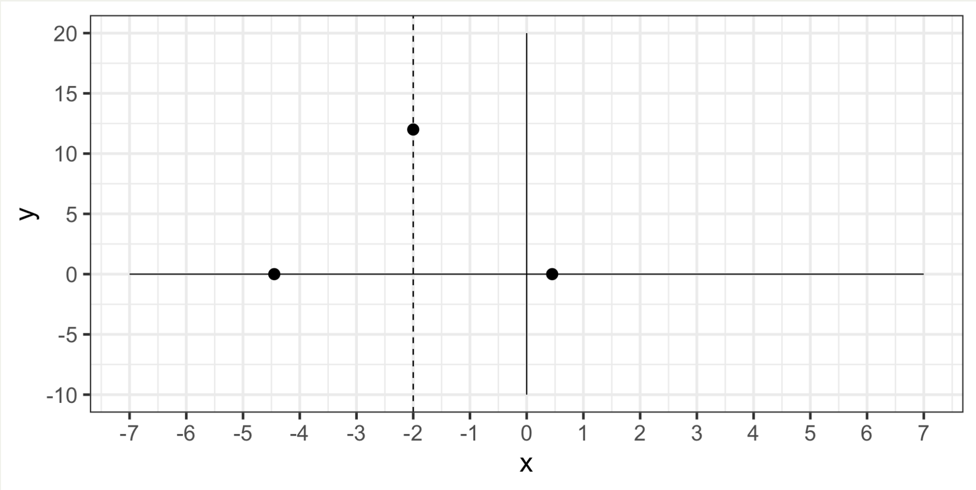
Solution.

Over the last several examples, we've investigated all of these items.

- The vertex is at the point $(-2, 12)$ and the axis of symmetry is at $x = -2$
- The roots are at the points $\left(-2 - \sqrt{6}, 0\right)$ and $\left(-2 + \sqrt{6}, 0\right)$



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- The vertex is at the point $(-2, 12)$ and the axis of symmetry is at $x = -2$
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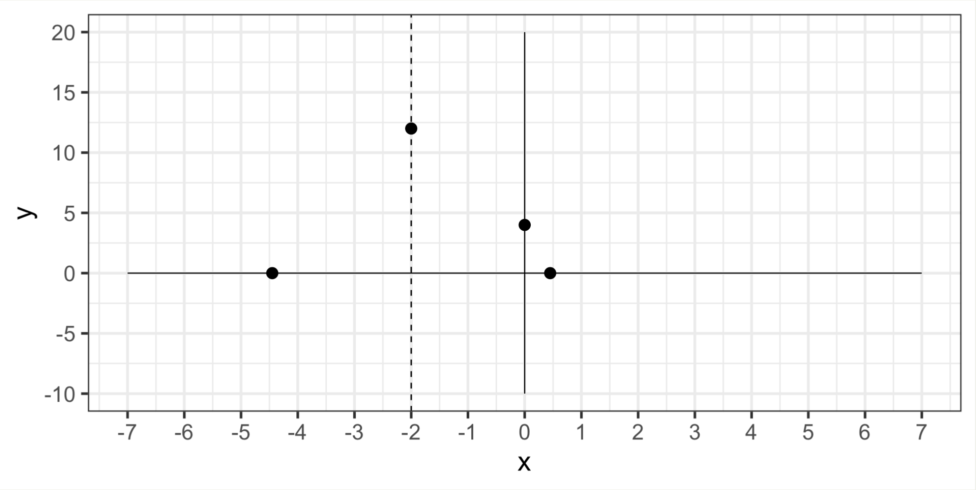

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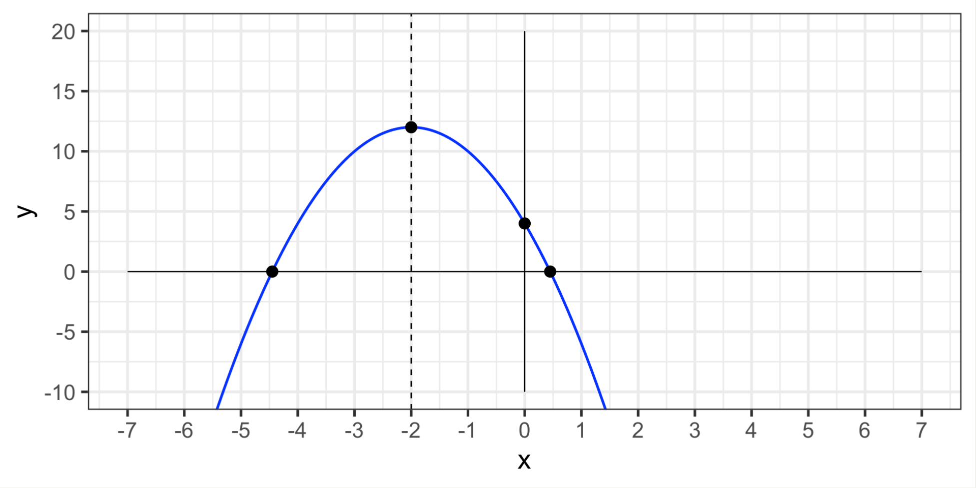
Solution.

Over the last several examples, we've investigated all of these items.

- The vertex is at the point $(-2, 12)$ and the axis of symmetry is at $x = -2$
- The roots are at the points $\left(-2 - \sqrt{6}, 0\right)$ and $\left(-2 + \sqrt{6}, 0\right)$
- The y-intercept is at the point $(0, 4)$
- The function is concave down
- The end behavior indicates $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = -\infty$
- The vertex is a maximum



Plotting a Quadratic Practice

Try It! 7: Use information about the concavity, roots, y-intercept, vertex, axis of symmetry, and end behavior of the function $f(x) = 24 - 2x - 2x^2$ to draw its graph.

Plotting a Quadratic Practice (Verified)

Try It! 7: Use information about the concavity, roots, y-intercept, vertex, axis of symmetry, and end behavior of the function $f(x) = 24 - 2x - 2x^2$ to draw its graph.

Solution. Rewriting the quadratic in standard form gives $f(x) = -2x^2 - 2x + 24$, so $a = -2$, $b = -2$, $c = 24$.

Concavity Since $a = -2 < 0$, the parabola opens **downward** (concave down). The vertex is a **maximum**.

Vertex: The x-coordinate of the vertex: $x = \frac{-b}{2a} = \frac{-(-2)}{2(-2)} = \frac{2}{-4} = -\frac{1}{2}$

Vertex: $\left(-\frac{1}{2}, \frac{49}{2}\right)$

The y-coordinate: $f\left(-\frac{1}{2}\right) = 24 - 2\left(-\frac{1}{2}\right) - 2\left(-\frac{1}{2}\right)^2 = 24 + 1 - \frac{1}{2} = \frac{49}{2}$

Roots. Solve $f(x) = 0$:

$$\begin{aligned} 24 - 2x - 2x^2 &= 0 \\ \implies -2\left(x^2 + x - 12\right) &= 0 \\ \implies (x+4)(x-3) &= 0 \\ \implies x = -4 \text{ or } x &= 3 \end{aligned}$$

Roots at $(-4, 0)$ and $(3, 0)$.

y-intercept: The y-intercept is at the point $(0, 24)$

End behavior. The leading term is $-2x^2$, so:

$$\lim_{x \to -\infty} f(x) = -\infty$$

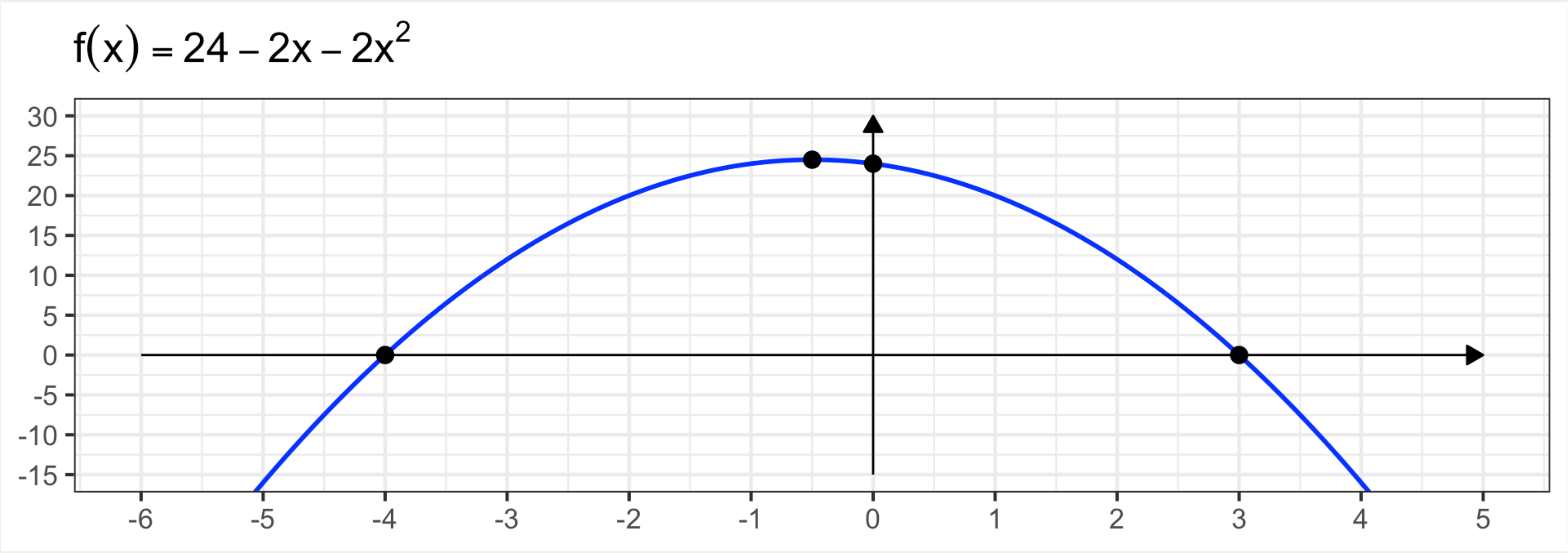
and

$$\lim_{x \to \infty} f(x) = -\infty$$

Plotting a Quadratic Practice (Verified)

Try It! 7: Use information about the concavity, roots, y-intercept, vertex, axis of symmetry, and end behavior of the function $f(x) = 24 - 2x - 2x^2$ to draw its graph.

(e) **Graph.** Using vertex $\left(-\frac{1}{2}, \frac{49}{2}\right)$, roots $(-4, 0)$ and $(3, 0)$, y-intercept $(0, 24)$, downward concavity, and end-behavior towards negative infinity:



Additional Practice

Try It! 8: For each quadratic function below, determine: (i) concavity, (ii) vertex, (iii) roots, (iv) end behavior, and (v) sketch a rough graph.

$(a)\;\; f(x) = x^2 - 4x - 5$

$(b)\;\; g(x) = -x^2 + 6x - 5$

Factoring versus the Quadratic Formula

For roots, try factoring first. If that doesn't work within a minute, use the quadratic formula.

Applied Problems

Applied Problem 1: A rocket is launched into the air. Its height (in meters) is $h\left(t\right)=-4.9t^{2}+229t+234$, where t is time in seconds.

Find the maximum height the rocket attains.

Solution. The maximum occurs at the vertex.

The quadratic function describing the height of the rocket has $a=-4.9$, $b=229$, and $c=234$.

$$t=\frac{-b}{2a}=\frac{-229}{2\left(-4.9\right)}=\frac{-229}{-9.8}\approx 23.37\text{ seconds}$$

We can plug the time at which the maximum height is reached (23.37 seconds) into the height function to determine the height of the rocket.

$$h\left(23.37\right)=-4.9(23.37)^{2}+229\left(23.37\right)+234\approx\boxed{2{,}909.6\text{ meters}}$$

Applied Problems (Cont'd)

Applied Problem 2: The price per unit (in dollars) for a cell phone is modeled by $p = 45 - 0.0125x$, where x is the number of phones produced (in thousands). The revenue (in thousands of dollars) is $R = x \cdot p$.

Find the production level that maximizes revenue.

Solution. First, write R as a function of x:

$$R(x) = x(45 - 0.0125x) = 45x - 0.0125x^2$$

This is a downward-opening parabola ($a = -0.0125 < 0$).

The maximum revenue occurs at the vertex:

$$x = \frac{-b}{2a} = \frac{-45}{2(-0.0125)} = \frac{-45}{-0.025} = \boxed{1{,}800 \text{ phones}}$$

The corresponding maximum revenue is $R(1800) = 45(1800) - 0.0125(1800)^2 = \boxed{\$40{,}500}$.

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `10. Quadratic Functions and Equations (Part I)`

Task: Consider the function $f(x) = x^2 - 2x - 8$.

(a) Find the roots by factoring.

(b) Find the vertex.

(c) Does the parabola open upward or downward?

Summary and Next Time...

Ideas From Today

- A **quadratic equation** has the form $ax^2 + bx + c = 0$ and may have 0, 1, or 2 real solutions.
- **Factoring** is fast when it works; use the **quadratic formula** when it doesn't.
- The **discriminant** $b^2 - 4ac$ tells you how many real solutions exist before you solve.
- The graph of a **quadratic function** $f(x) = ax^2 + bx + c$ forms a parabola.
- **Concavity** is determined by the sign of a; the **vertex** is at $x = \frac{-b}{2a}$.
- The **end behavior** is determined by the leading term ax^2.

Looking Ahead

- Next class we'll develop **completing the square** as a third technique for solving quadratics – and as the key to writing quadratic functions in **vertex form**, which connects directly to the transformations framework.

Next Time:
Quadratic Functions and Equations (Part II)
Homework:
Start Homework 8 on MyOpenMath