MAT 142: Complex Numbers

Dr. Gilbert

June 3, 2026

Reminders

At our last meeting, we discussed each of the following:

- Linear functions
  - The slope of a line
  - The y-intercept of a line
  - Writing the equation of a line in slope-intercept form
  - Writing the equation of a line in point-slope form
- Connecting linear functions to the graphical transformation framework
- Parallel and perpendicular lines
- Solving linear equations and the corresponding geometric interpretation

Try the following warm-up problems.

Problem 1: Simplify the expression $(2 + 3x) + (7 - 2x)$.

Problem 2: Simplify the expression $(2 + 3x)(7 - 2x)$.

Note: These warm-up problems are designed to preview today's topic. Keep your answers in mind as we move forward.

Objectives

Today we take a brief but important detour to introduce the *complex numbers* before moving into our study of polynomial functions.

After today's class meeting, you should be able to:

- Write and recognize complex numbers of the form $a + bi$, where a, b are real numbers and $i = \sqrt{(-1)}$.
- Identify and construct the complex conjugate of a complex number.
- Add, subtract, multiply, and divide complex numbers, expressing all results in the form $a + bi$.

What Are Complex Numbers?

Definition (Complex Number): A *complex number* is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{(-1)}$.

The collection of all complex numbers is denoted $\mathbb{C}$.

- a is called the *real part* and b is called the *imaginary part*.
- Real numbers are complex numbers with $b = 0$. So $\mathbb{R}$ is a subset of $\mathbb{C}$ (or, in symbols, $\mathbb{R} \subset \mathbb{C}$).
- Pure imaginary numbers have $a = 0$, such as $3i$ or $-7i$.

Powers of i:

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 \qquad i^5 = i \qquad \cdots$$

The pattern repeats with period 4. This is the key fact that makes complex arithmetic work.

Adding and Subtracting Complex Numbers

Strategy: Adding and subtracting complex numbers works just like combining like terms with a variable. Treat i as if it were a variable such as x.

Combine real parts with real parts, and imaginary parts with imaginary parts.

Example 1: Compute the sum $(5 - 7i) + (3i)$.

Solution. Combine like terms:

$$(5 - 7i) + (3i)$$

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$$(5 - 7i) + (3i) = 5 + (-7i + 3i)$$

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Solution. Combine like terms:

$$\begin{aligned}(5 - 7i) + (3i) &= 5 + (-7i + 3i) \\ &= \boxed{5 - 4i}\end{aligned}$$

Example 2: Compute the difference $(1 + 18i) - (-3 - i)$.

Solution.

$$(1 + 18i) - (-3 - i)$$

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$$(1 + 18i) - (-3 - i) = 1 + 18i + 3 + i$$

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$$\begin{aligned}(1 + 18i) - (-3 - i) &= 1 + 18i + 3 + i \\ &= (1 + 3) + (18i + i)\end{aligned}$$

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Solution.

$$\begin{aligned}(1 + 18i) - (-3 - i) &= 1 + 18i + 3 + i \\ &= (1 + 3) + (18i + i) \\ &= \boxed{4 + 19i}\end{aligned}$$

Adding and Subtracting Practice

Try It! 1: Simplify $(3+5i)+(2-9i)$

Try It! 2: Simplify $(4-i)-(-1+6i)$

Multiplying Complex Numbers

Strategy: Multiply complex numbers just like polynomial expressions – distribute (*multiply everything in one factor by everything in the other factor*), then simplify using the fact that $i^2 = -1$ to eliminate any i^2 terms and express the result in the form $a + bi$.

Example 3: Simplify $(6 + 2i)(-4 - 3i)$.

Solution.

$$(6 + 2i)(-4 - 3i)$$

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Example 3: Simplify $(6 + 2i)(-4 - 3i)$.

Solution.

$$(6 + 2i)(-4 - 3i) = -24 - 18i - 8i - 6i^2$$

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$$\begin{aligned}(6 + 2i)(-4 - 3i) &= -24 - 18i - 8i - 6i^2 \\ &= -24 - 26i - 6(-1)\end{aligned}$$

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$$\begin{aligned}(6 + 2i)(-4 - 3i) &= -24 - 18i - 8i - 6i^2 \\ &= -24 - 26i - 6(-1) \\ &= -24 - 26i + 6\end{aligned}$$

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Example 3: Simplify $(6+2i)(-4-3i)$.

Solution.

$$\begin{aligned} (6+2i)(-4-3i) &= -24 - 18i - 8i - 6i^2 \\ &= -24 - 26i - 6(-1) \\ &= -24 - 26i + 6 \\ &= \boxed{-18 - 26i} \end{aligned}$$

Example 4: Simplify $(5-8i)(5+8i)$.

Solution.

$$(5-8i)(5+8i)$$

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Example 4: Simplify $(5 - 8i)(5 + 8i)$.

Solution.

$$(5-8i)(5+8i) = 25 + 40i - 40i - 64i^2$$

Multiplying Complex Numbers

Strategy: Multiply complex numbers just like polynomial expressions – distribute (*multiply everything in one factor by everything in the other factor*), then simplify using the fact that $i^2 = -1$ to eliminate any i^2 terms and express the result in the form $a + bi$.

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Solution.

$$\begin{aligned}(5 - 8i)(5 + 8i) &= 25 + 40i - 40i - 64i^2 \\ &= 25 - 64(-1) \\ &= 25 + 64 \\ &= \boxed{89}\end{aligned}$$

The result is a *real* number. What was special about the two complex numbers we multiplied?

Multiplication Practice

Try It! 3: Simplify $(2+7i)\,(3-4i)$

Try It! 4: Simplify $(1-3i)\,(1+3i)$

Try It! 5: Simplify $(9-2i)\,(9+2i)$

Complex Conjugates

Definition (Complex Conjugate): The *complex conjugate* of the complex number $a + bi$ is the complex number $a - bi$.

In **Example 4**, the numbers $5 - 8i$ and $5 + 8i$ are *complex conjugates* of one another. Their product was the real number 89. This happened in **Try It! 5** for the same reason.

Products of Complex Conjugates

The product of a complex number and its conjugate is *always* a real number.

$$(a + bi)\,(a - bi)$$

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> **Products of Complex Conjugates**
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> The product of a complex number and its conjugate is *always* a real number.

$$(a + bi)\,(a - bi) = a^2 - abi + abi - b^2 i^2$$

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Products of Complex Conjugates

The product of a complex number and its conjugate is *always* a real number.

$$\begin{aligned}(a+bi)(a-bi) &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= \boxed{a^2 + b^2}\end{aligned}$$

This fact is what makes division of complex numbers possible.

Division

Strategy: To divide complex numbers, multiply both the numerator and denominator by the *complex conjugate of the denominator*. This makes the denominator real, allowing us to write the result in the form $a + bi$.

Example 5: Simplify $\dfrac{5 - 2i}{1 + i}$.

Solution. Multiply numerator and denominator by the conjugate of the denominator, $1 - i$.

$$\frac{5 - 2i}{1 + i}$$

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Solution. Multiply numerator and denominator by the conjugate of the denominator, $1 - i$.

$$\frac{5-2i}{1+i} = \left(\frac{5-2i}{1+i}\right) \cdot \left(\frac{1-i}{1-i}\right)$$

Division

Strategy: To divide complex numbers, multiply both the numerator and denominator by the *complex conjugate of the denominator*. This makes the denominator real, allowing us to write the result in the form $\boldsymbol{a + bi}$.

Example 5: Simplify $\dfrac{5-2i}{1+i}$.

Solution. Multiply numerator and denominator by the conjugate of the denominator, $\mathbf{1} - \boldsymbol{i}$.

$$\begin{aligned}\frac{5-2i}{1+i} &= \left(\frac{5-2i}{1+i}\right)\cdot\left(\frac{1-i}{1-i}\right)\\ &= \frac{(5-2i)\,(1-i)}{(1+i)\,(1-i)}\end{aligned}$$

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$$\begin{aligned}\frac{5-2i}{1+i} &= \left(\frac{5-2i}{1+i}\right)\cdot\left(\frac{1-i}{1-i}\right)\\ &= \frac{(5-2i)(1-i)}{(1+i)(1-i)}\\ &= \frac{5-5i-2i+2i^2}{1-i+i-i^2}\end{aligned}$$

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$$\begin{aligned}
\frac{5-2i}{1+i} &= \left(\frac{5-2i}{1+i}\right) \cdot \left(\frac{1-i}{1-i}\right) \\
&= \frac{(5-2i)(1-i)}{(1+i)(1-i)} \\
&= \frac{5-5i-2i+2i^2}{1-i+i-i^2} \\
&= \frac{5-7i+2(-1)}{1-(-1)}
\end{aligned}$$

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&= \frac{5-5i-2i+2i^2}{1-i+i-i^2}\\
&= \frac{5-7i+2(-1)}{1-(-1)}\\
&= \frac{3-7i}{2}
\end{aligned}$$

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&= \frac{5-5i-2i+2i^2}{1-i+i-i^2} \\
&= \frac{5-7i+2(-1)}{1-(-1)} \\
&= \frac{3-7i}{2} \\
&= \boxed{\frac{3}{2} - \frac{7}{2}i}
\end{aligned}$$

Division

Example 6: Simplify $\dfrac{-2-5i}{3i}$.

Solution. Note that $3i = 0 + 3i$, so its complex conjugate is $0 - 3i = -3i$.

$$\frac{-2-5i}{3i}$$

Division

Example 6: Simplify $\dfrac{-2-5i}{3i}$.

Solution. Note that $3i = 0 + 3i$, so its complex conjugate is $0 - 3i = -3i$.

$$\frac{-2-5i}{3i} = \left(\frac{-2-5i}{3i}\right) \cdot \left(\frac{-3i}{-3i}\right)$$

Division

Example 6: Simplify $\dfrac{-2-5i}{3i}$.

Solution. Note that $3i = 0 + 3i$, so its complex conjugate is $0 - 3i = -3i$.

$$\begin{aligned}\frac{-2-5i}{3i} &= \left(\frac{-2-5i}{3i}\right)\cdot\left(\frac{-3i}{-3i}\right)\\ &= \frac{(-2-5i)(-3i)}{(3i)(-3i)}\end{aligned}$$

Division

Example 6: Simplify $\dfrac{-2-5i}{3i}$.

Solution. Note that $3i = 0 + 3i$, so its complex conjugate is $0 - 3i = -3i$.

$$\begin{aligned}
\frac{-2-5i}{3i} &= \left(\frac{-2-5i}{3i}\right) \cdot \left(\frac{-3i}{-3i}\right) \\
&= \frac{(-2-5i)\,(-3i)}{(3i)\,(-3i)} \\
&= \frac{6i+15i^2}{-9i^2}
\end{aligned}$$

Division

Example 6: Simplify $\dfrac{-2-5i}{3i}$.

Solution. Note that $3i = 0 + 3i$, so its complex conjugate is $0 - 3i = -3i$.

$$\begin{aligned}
\frac{-2-5i}{3i} &= \left(\frac{-2-5i}{3i}\right) \cdot \left(\frac{-3i}{-3i}\right) \\
&= \frac{(-2-5i)(-3i)}{(3i)(-3i)} \\
&= \frac{6i + 15i^2}{-9i^2} \\
&= \frac{6i + 15(-1)}{-9(-1)}
\end{aligned}$$

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&= \frac{6i+15i^2}{-9i^2} \\
&= \frac{6i+15(-1)}{-9(-1)} \\
&= \frac{6i-15}{9}
\end{aligned}$$

Division

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&= \frac{6i+15i^2}{-9i^2} \\
&= \frac{6i+15(-1)}{-9(-1)} \\
&= \frac{6i-15}{9} \\
&= \frac{-15}{9} + \frac{6}{9}i
\end{aligned}$$

Division

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\frac{-2-5i}{3i} &= \left(\frac{-2-5i}{3i}\right)\cdot\left(\frac{-3i}{-3i}\right)\\
&= \frac{(-2-5i)(-3i)}{(3i)(-3i)}\\
&= \frac{6i+15i^2}{-9i^2}\\
&= \frac{6i+15(-1)}{-9(-1)}\\
&= \frac{6i-15}{9}\\
&= \frac{-15}{9}+\frac{6}{9}i\\
&= \boxed{-\frac{5}{3}+\frac{2}{3}i}
\end{aligned}$$

Division Practice

Try It! 6: Simplify $\dfrac{3+i}{2-i}$

Try It! 7: Simplify $\dfrac{-7i}{3+4i}$

Try It! 8: Simplify $\dfrac{3+4i}{-7i}$

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `9. Complex Numbers`

Task: Simplify $(2+5i)(1-6i)$. Express your answer in the form $a+bi$.

Summary and Next Time...

Ideas From Today

- A **complex number** has the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
- Real numbers are complex numbers with $b = 0$, so $\mathbb{R}$ is a subset of $\mathbb{C}$.
- **Add/subtract:** combine real parts and imaginary parts separately (as *like terms*).
- **Multiply:** distribute and use $i^2 = -1$ to simplify.
- The **complex conjugate** of $a + bi$ is $a - bi$; the product $(a + bi)(a - bi)$ is always real: $a^2 + b^2$.
- **Divide:** multiply numerator and denominator by the conjugate of the denominator, then simplify.

Looking Ahead

- Complex numbers arise naturally when solving quadratic equations – the quadratic formula can produce square roots of negative numbers, and complex numbers are exactly what we need to make sense of them.
- When we study quadratic (and higher-degree polynomial) functions next, we'll see that complex numbers complete the picture of what roots can look like.

Next Time:
Quadratic Functions and Equations
Homework:
Complete Homework 7 on MyOpenMath