MAT 142: Linear Functions and Equations

Dr. Gilbert

June 3, 2026

Reminders

At our last meeting, we discussed each of the following:

- The definition of inverse functions and verification via composition.
- The horizontal line test for invertibility.
- The swap-and-solve method for finding $f^{-1}(x)$.
- The geometric symmetry of a function and its inverse across $y = x$.

Try the following warm-up problem.

Problem 1: Find the domain, range, y-intercept, roots, and the inverse (if it exists) for the function $f(x) = 5x - 15$.

Reminders:

- The *domain* of $f(x)$ is the set of all permissible inputs.
- The *range* of $f(x)$ is the set of all attainable outputs.
  - Use what you know about the “book function” $f(x) = x$ and transformations to find the range.
- The y-intercept is the point $(0, f(0))$.
- The *roots* are all solutions to $f(x) = 0$, which are of the form $(x, 0)$.
- Invertible functions pass the *horizontal line test*; use swap-and-solve to find f^{-1}.

Motivating Application

A gym membership with two personal training sessions costs \$125. A gym membership with five personal training sessions costs \$260.

What is the cost per session? The gym would like to offer a discounted rate for members signing up for ten sessions – they'll take \$5 off the cost for *each* session. How much would that membership cost?

Before we work through this, consider the following:

- What does using a linear model assume about this scenario?
- Can you think of alternative discount strategies the gym could offer? Would those still result in a linear model?

We'll return to this application at the end of today's class.

Objectives

Today's class has two connected halves.

First half – Linear Functions:

- Recognize functions of the form $f(x) = mx + b$ as linear functions.
- Identify the slope m and y-intercept $(0, b)$.
- Use the slope formula and point-slope form to construct the equation of a line.
- Connect linear functions to the transformations framework from Day 8.
- Determine whether a linear function is increasing, decreasing, or constant.

Second half – Linear Equations:

- Solve linear equations in a single variable.
- Interpret solving equations geometrically as finding intersections or roots.

Linear Functions

Definition (Linear Function): A function $f(x)$ is *linear* if it can be written in the form

$$f(x) = mx + b$$

where m is the *slope* and $(0, b)$ is the *y-intercept*.

The slope m measures the rate of change of the function: for every one-unit increase in x, the output changes by exactly m units.

A linear function is:

- **Increasing** if $m > 0$
- **Decreasing** if $m < 0$
- **Constant** if $m = 0$

Forms for the Equation of a Line

There are several forms for representing a line. The most useful are:

Name	**Form**	**Best Use**
Slope-Intercept	$y = mx + b$	Reading slope and y-intercept directly
Point-Slope	$y - y_0 = m\left(x - x_0\right)$	Constructing the equation of a line
Standard	$Ax + By = C$	Generalizes cleanly to higher dimensions

We'll focus on **slope-intercept** and **point-slope** forms, since they're the ones with the greatest practical uses in both PreCalculus and Calculus.

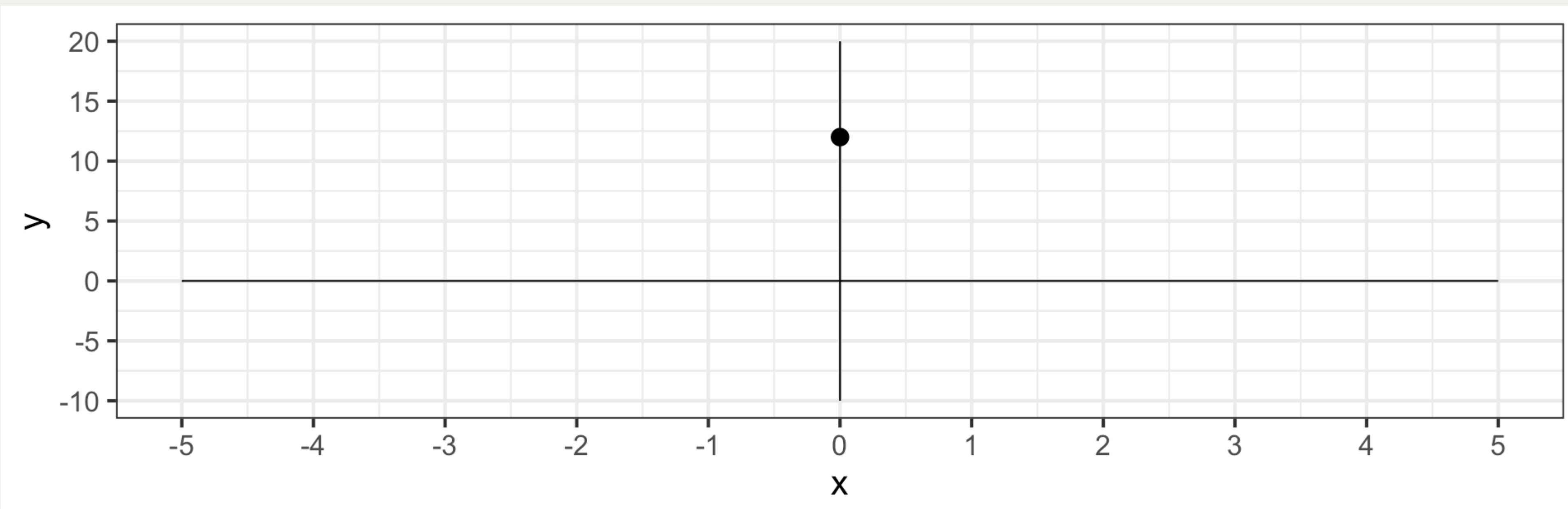
Slope and Intercept

Example 1: Determine the slope and y-intercept of $y = -8x + 12$. Once you're done, draw a graph of the function $f(x) = -8x + 12$.

Solution. This is already in slope-intercept form.

- Slope: $m = \boxed{-8}$
- y-intercept: $\boxed{(0, 12)}$

Since $m = -8$, we have $m < 0$ and this is a **decreasing** linear function.



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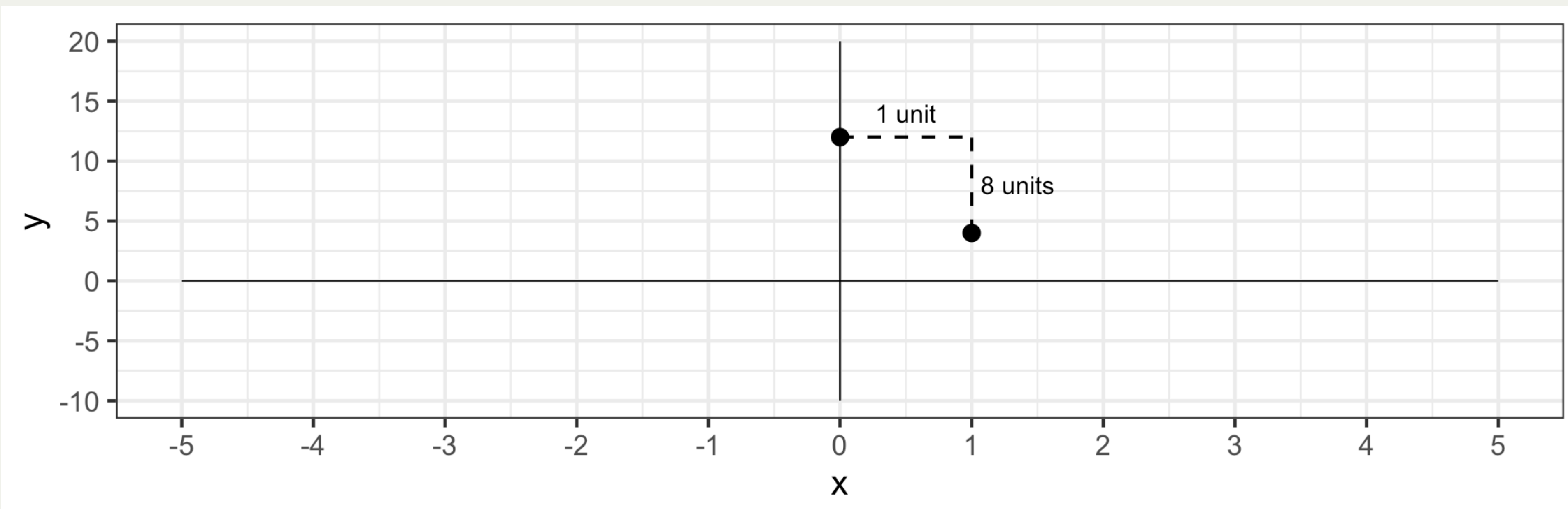

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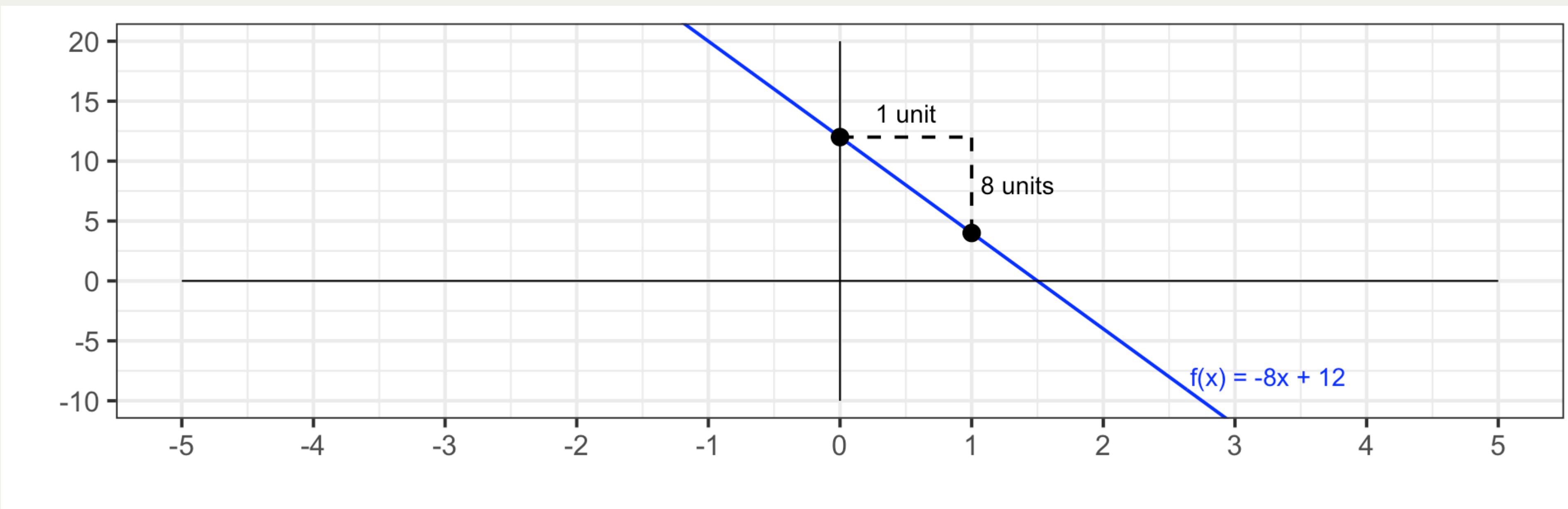

Slope and Intercept

Example 2: Find the equation of the line with slope $m = \frac{1}{2}$ passing through $(-2, 5)$. Write it in slope-intercept form and graph the function.

Solution. Since we know $m = \frac{1}{2}$, start with $y = \frac{1}{2}x + b$ and substitute the known point to solve for b.

$$5 = \frac{1}{2}(-2) + b$$

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$$\begin{aligned} 5 &= \frac{1}{2}(-2) + b \\ \implies 5 &= -1 + b \end{aligned}$$

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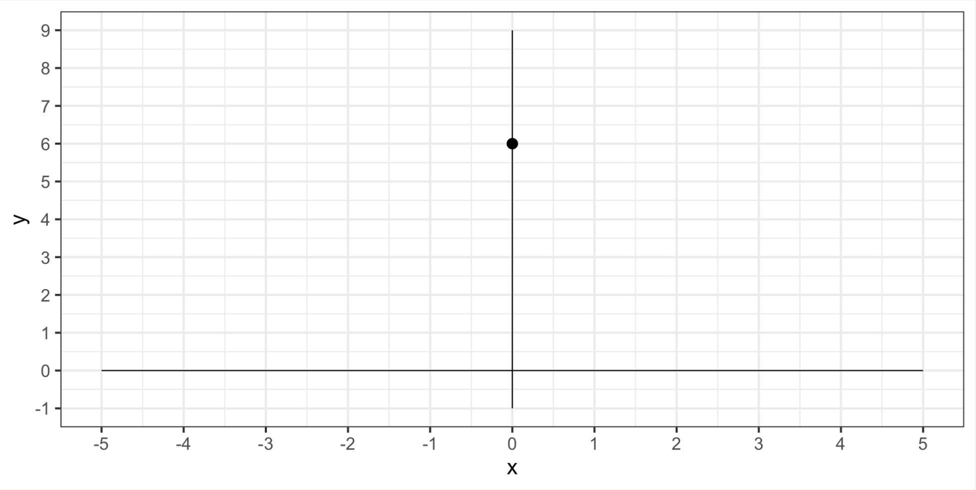
$$\begin{aligned} 5 &= \frac{1}{2}(-2) + b \\ \implies 5 &= -1 + b \\ \implies b &= 6 \end{aligned}$$

So the equation is $\boxed{y = \frac{1}{2}x + 6}$.



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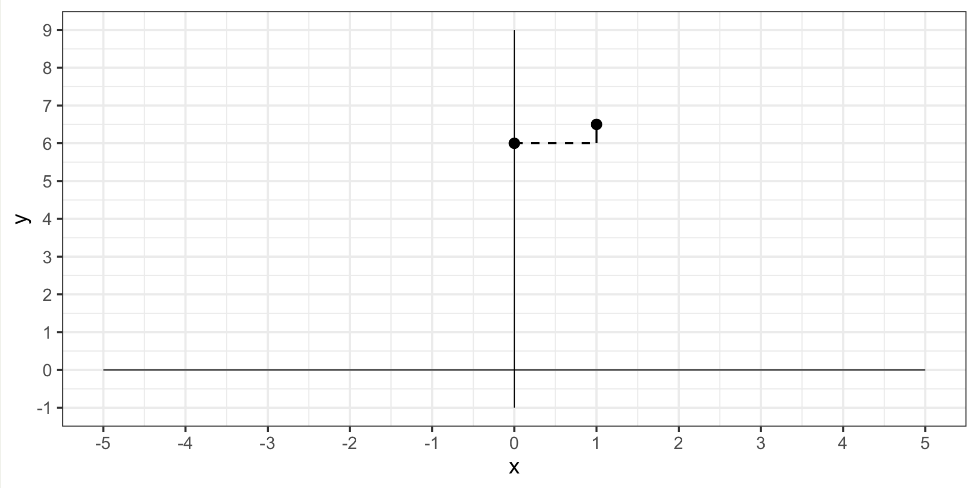

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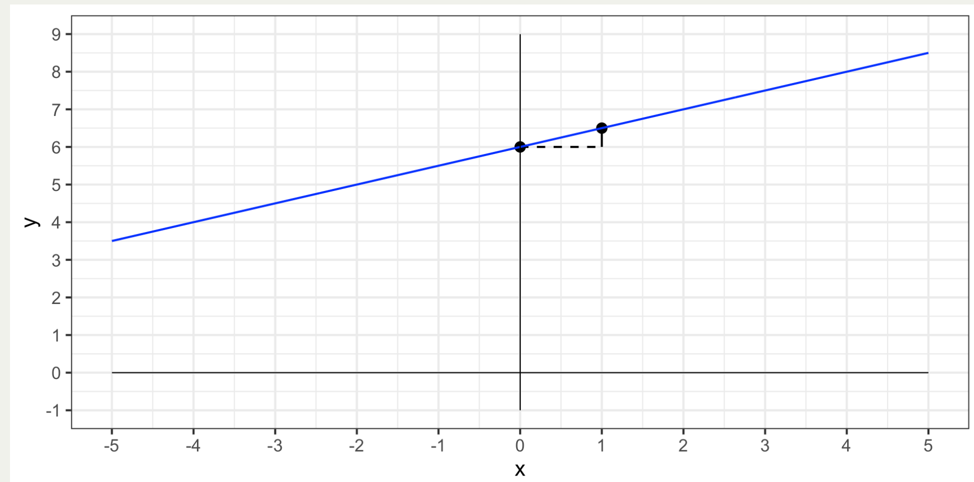

Use Point-Slope Form to Build Equations Instead

The approach on this slide works, but there is a more efficient, and less error proned, approach using *point-slope* form. We'll revisit this example next.

Point-Slope Form

Definition (Point-Slope Form): Given a line with slope m passing through the point (x_0, y_0), the equation of the line is:

$$y - y_0 = m\,(x - x_0)$$

This is the most efficient form when constructing the equation of a line – you need only a slope and one point.

Example 3: Revisiting Example 2 – find the equation of the line with slope $\dfrac{1}{2}$ through $(-2, 5)$ using point-slope form.

Solution. Substitute directly:

$$\boxed{y - 5 = \frac{1}{2}(x - (-2))}$$

To convert to slope-intercept form:

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$$\begin{aligned} & y - 5 = \frac{1}{2}(x - (-2)) \\ \implies & y - 5 = \frac{1}{2}(x + 2) \\ \implies & y - 5 = \frac{1}{2}x + 1 \end{aligned}$$

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$$\begin{aligned} y - 5 &= \frac{1}{2}(x - (-2)) \\ \implies y - 5 &= \frac{1}{2}(x + 2) \\ \implies y - 5 &= \frac{1}{2}x + 1 \\ \implies y &= \frac{1}{2}x + 6 \checkmark \end{aligned}$$

Point Slope Form

Example 4: Find the equation of the line passing through $(3, 8)$ and $(7, -4)$.

Solution. First compute the slope using the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point Slope Form

Example 4: Find the equation of the line passing through $(3, 8)$ and $(7, -4)$.

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$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - 8}{7 - 3} \end{aligned}$$

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$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - 8}{7 - 3} \\ &= \frac{-12}{4} = -3 \end{aligned}$$

Now use point-slope form with $m = -3$ and the point $(3, 8)$:

$$\boxed{y - 8 = -3\,(x - 3)}$$

Note: Either point works – using $(7, -4)$ gives $y + 4 = -3\,(x - 7)$, which simplifies to the same line.

Linear Functions and Transformations

Recall from Day 8 that every function class is built by transforming a *book function*. For linear functions, that book function is $f(x) = x$.

Consider the function from Example 4 written as $f(x) = -\mathbf{3}(x - \mathbf{3}) + 8$.

This makes the transformation sequence explicit:

1. Start with $g(x) = x$
2. Shift **right** by $\mathbf{3}$, which is done via $g(x - \mathbf{3}) = x - \mathbf{3}$
3. Reflect over the x-axis and stretch by $\mathbf{3}$ to get $-\mathbf{3} \cdot g(x - \mathbf{3}) = -\mathbf{3}(x - \mathbf{3})$
4. Shift **up** by 8, obtaining $\mathbf{3} \cdot g(x - \mathbf{3}) + 8 = -\mathbf{3}(x - \mathbf{3}) + 8$

This is exactly the point-slope form $y - 8 = -\mathbf{3}(x - \mathbf{3})$ – written as a function! Point-slope form and the transformations framework are two descriptions of the same idea.

Vertical and Horizontal Lines

A line is *horizontal* if its slope is 0.

The equation for a horizontal line simplifies to $y = b$, since $y = 0x + b$ is the same as $y = b$.

Vertical lines have undefined slopes, with equations written as $x = a$

Example 5: Find the equation for the *horizontal* line passing through the point $(5, -12)$.

Solution. The equation will have the form $y = b$. This means that the y-coordinate (the second coordinate) of all points on the line must remain constant at -12, so the equation must be $y = -12$.

Example 6: Find the equation of the *vertical* line passing through the point $(4, 18)$.

Solution. This time, the equation will take the form $x = a$ and the x-coordinate of every point on the line must remain constant. Since the known point has an x-coordinate at 4, we know that the equation of the line must be $x = 4$.

Parallel and Perpendicular Lines

Definition: Two lines are **parallel** if they have the same slope. Two lines are **perpendicular** if their slopes are negative reciprocals of one another. That is, $m_1 \cdot m_2 = -1$.

Example 7: Find any line parallel to $y = -17x + 11$.

Solution. Any line with slope -17 will do. For example, $\boxed{y = -17x - 1}$.

Example 8: Find the equation of the line perpendicular to $y = 3x + 8$ and passing through $(-5, 12)$.

Solution. The perpendicular slope is $m = -\frac{1}{3}$ (negative reciprocal of 3). Using point-slope form:

$$\boxed{y - 12 = -\frac{1}{3}(x + 5)}$$

Equations of Lines Practice

Try It! 1: Find the equation of the line passing through $(-1, 4)$ and $(3, -8)$.

- Write the equation in both point-slope and slope-intercept forms.
- Is this line increasing or decreasing?

Try It! 2: Determine whether the following pairs of lines are parallel, perpendicular, or neither.

(a) $y = 4x - 7$ and $y = 4x + 2$

(b) $y = \frac{2}{3}x + 1$ and $y = -\frac{3}{2}x - 5$

(c) $y = 5x + 3$ and $y = -5x + 3$

Linear Equations

Definition (Linear Equation): An equation is *linear* in its variable(s) if it involves no exponents on variables, no variables in denominators, no products between variables, and no special functions.

Examples of linear equations:

- $5x - 4 = 22$
- $10x + 6 = -2\left(x - 5\right)$
- $2x + 3y = 19 + x$

Strategy (Solving Linear Equations):

1. Collect all variable terms on one side.
2. Move all constant terms to the other side.
3. Combine like terms.
4. Divide to isolate the variable.

Solving Linear Equations

Example 9: Solve $5x - 20 = 15$.

Solution.

$$5x - 20 = 15$$

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Solution.

$$\begin{aligned} 5x - 20 &= 15 \\ \implies 5x &= 35 \end{aligned}$$

Solving Linear Equations

Problem: Solve $5x - 20 = 15$.

Solution.

$$\begin{aligned} 5x - 20 &= 15 \\ \implies 5x &= 35 \\ \implies x &= \boxed{7} \end{aligned}$$

Example 10: Solve $2t + 18 = \dfrac{t+2}{3} + 5$.

Solution.

$$2t + 18 = \frac{t+2}{3} + 5$$

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$$\begin{aligned} 2t + 18 &= \frac{t+2}{3} + 5 \\ \implies 2t + 13 &= \frac{t+2}{3} \end{aligned}$$

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5x - 20 &= 15 \\
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\begin{aligned}
2t + 18 &= \frac{t+2}{3} + 5 \\
\implies 2t + 13 &= \frac{t+2}{3} \\
\implies 6t + 39 &= t + 2
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The Geometry of Solving Equations

Every algebraic manipulation we make when solving an equation has a geometric interpretation.

- The solutions to $\mathbf{expr}_1 = \mathbf{expr}_2$ are the *x-coordinates of any points of intersection* between the graphs of $f\left(x\right) = \mathbf{expr}_1$ and $g\left(x\right) = \mathbf{expr}_2$.
- The solutions to $\mathbf{expr} = \mathbf{0}$ are the *roots* (x-intercepts) of the graph of $f\left(x\right) = \mathbf{expr}$.

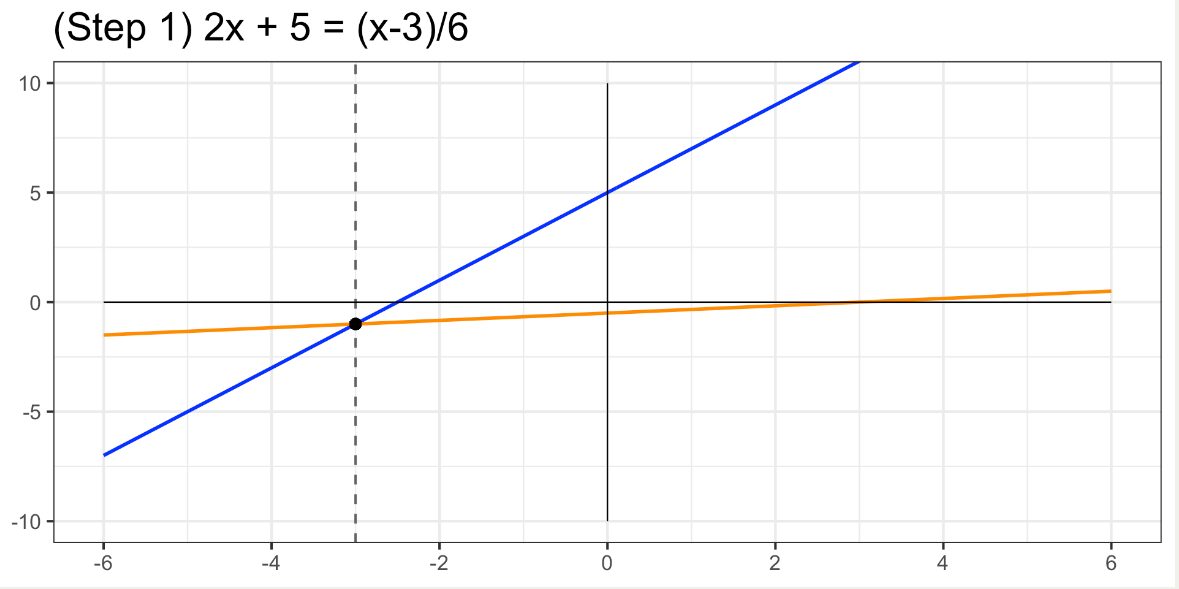
Each legal algebraic step changes the *form* of the equation – and with it, the *shape* of the geometric question – but never changes the *location* of the solution.

Geometry of Solving Equations

Example 11: Solve $2x + 5 = \dfrac{x-3}{6}$ and illustrate each step geometrically.

Solution.

$$2x + 5 = \frac{x-3}{6} \qquad \text{(Step 1)}$$

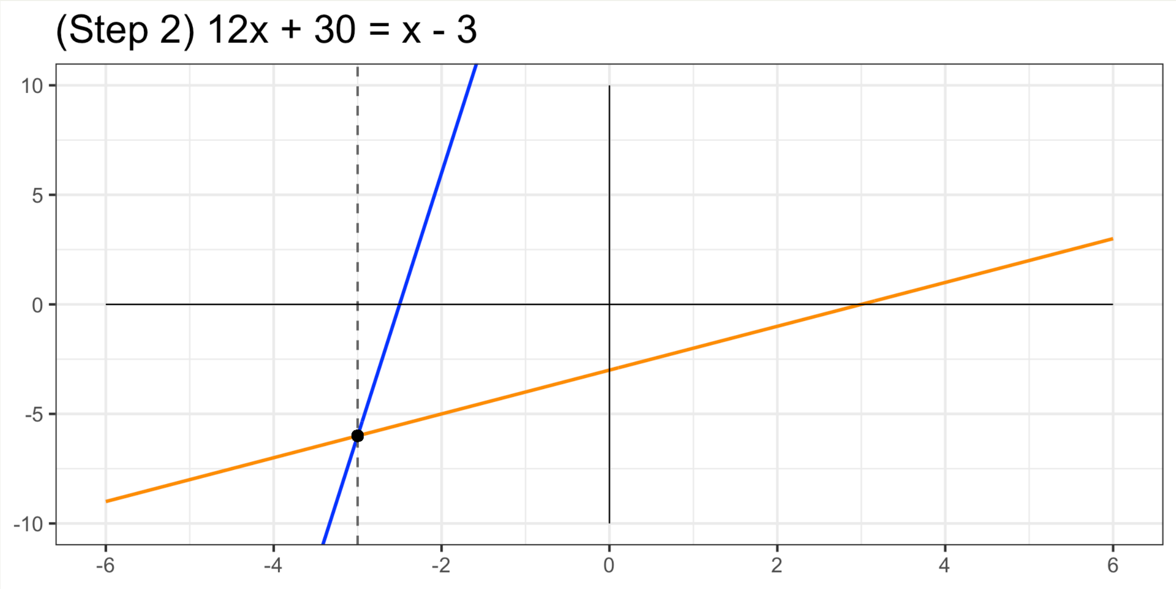


Geometry of Solving Equations

Example 11: Solve $2x + 5 = \dfrac{x-3}{6}$ and illustrate each step geometrically.

Solution.

$$\begin{aligned} 2x + 5 &= \frac{x-3}{6} && \text{(Step 1)} \\ \implies 12x + 30 &= x - 3 && \text{(Step 2)} \end{aligned}$$



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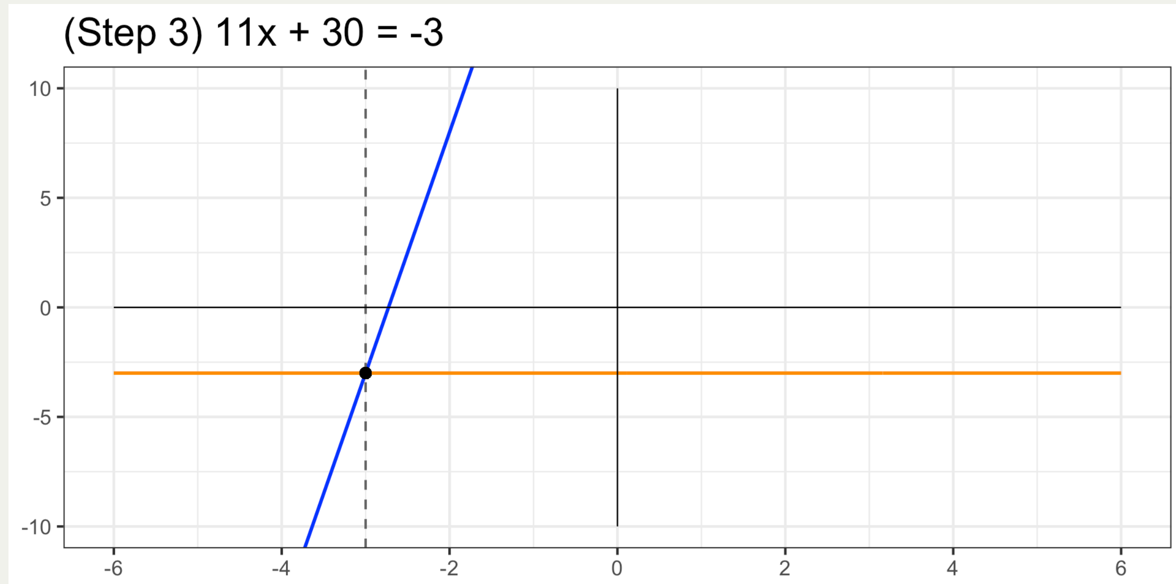

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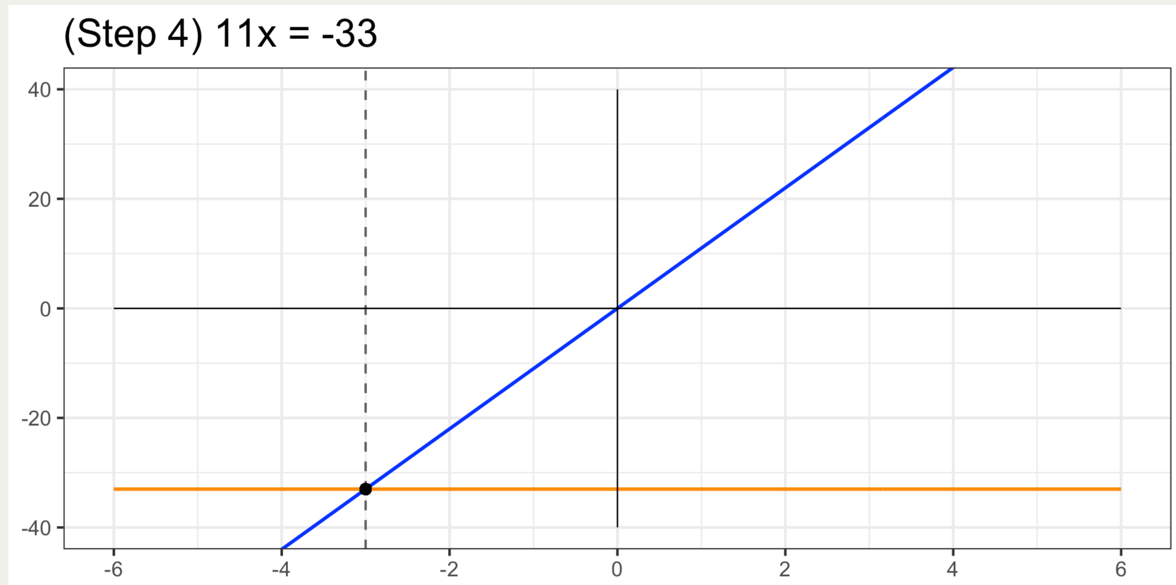

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\implies 11x + 30 &= -3 && \text{(Step 3)}\\
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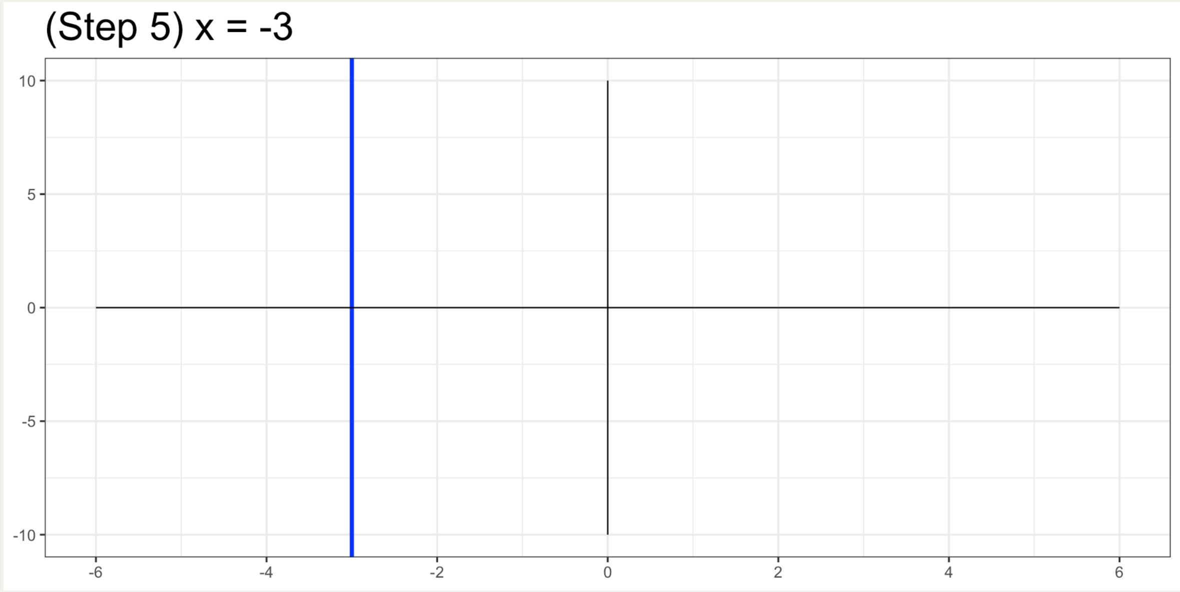
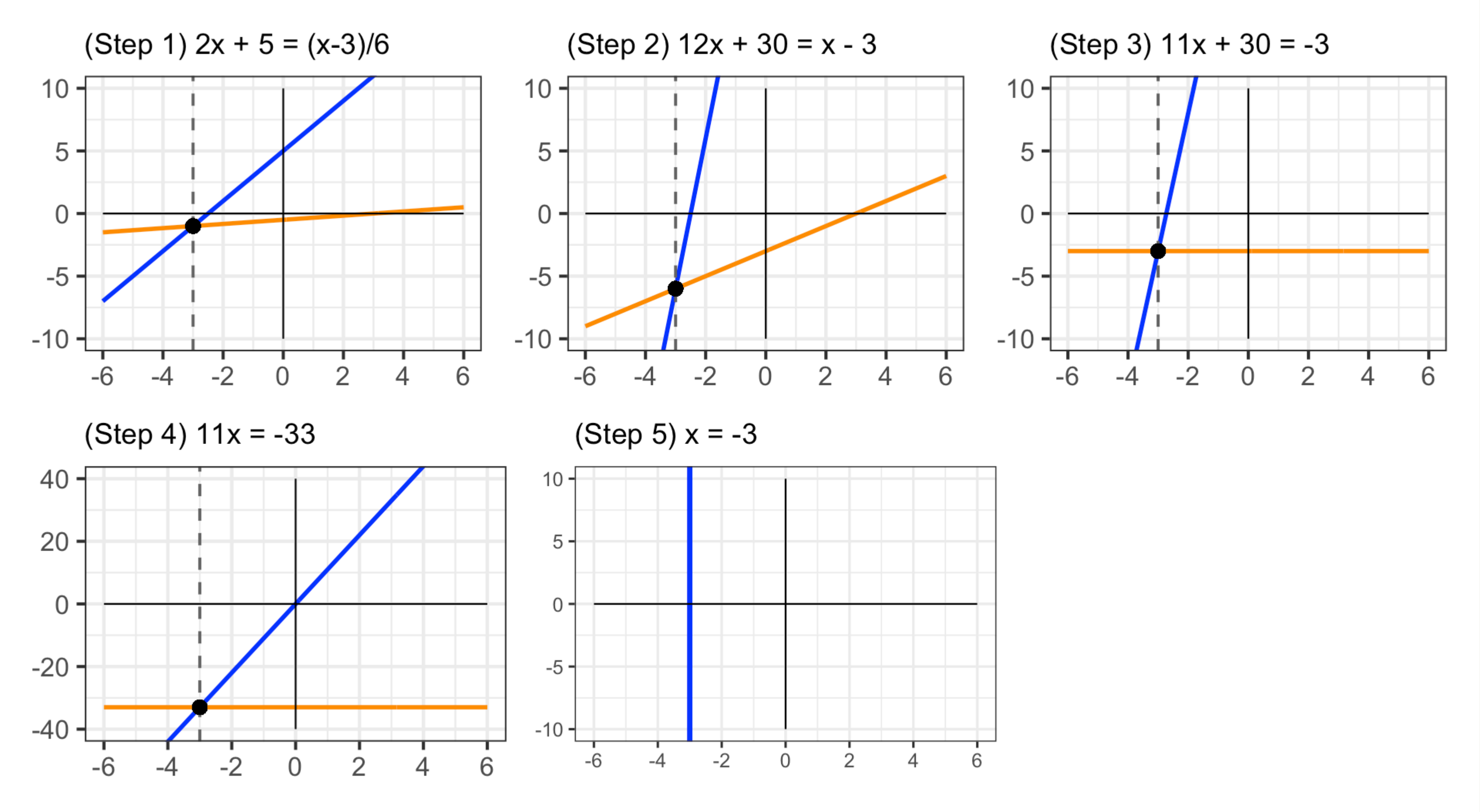
\implies x &= -3 && \text{(Step 5)}
\end{aligned}
$$



Geometry of Solving Equations

Below are all the graphical representations of the steps to solve the equation.



Applied Problems

Applied Problem 1: A gym membership with 2 personal training sessions costs \$125; with 5 sessions it costs \$260. What is the cost per session? If the gym offers \$5 off per session for members signing up for 10 sessions, how much would that membership cost?

Solution. Model the cost as $C(n) = mn + b$ where n is the number of sessions.

$$m = \frac{260 - 125}{5 - 2} = \frac{135}{3} = \$45 \text{ per session}$$

Solve for b using $C(2) = 125$: $\quad 125 = 45(2) + b \implies b = \$35$

So $C(n) = 45n + 35$. For the discounted 10-session package, the rate drops to \$40 per session:

$$C_{\text{discounted}}(10) = 40(10) + 35 = \boxed{\$435}$$

Applied Problems (Cont'd)

Applied Problem 2: A town's population has been growing linearly. In 2003 the population was $45{,}000$, and has been growing by $1{,}700$ people per year. Write a function $P(t)$ for the population t years after 2003.

Solution. We know the slope ($m = 1700$) and a point ($t = 0$, $P = 45000$), so:

$$\boxed{P(t) = 1700t + 45000}$$

Applied Problem 3: Average annual income for 1990–1999 is modeled by $I(x) = 1054x + 23286$, where x is years after 1990. Which statement correctly interprets the slope?

1. As of 1990, average annual income was $23,286.
2. Over the ten-year period, income increased by a total of $1,054.
3. Each year in the 1990s, average annual income increased by $1,054.
4. Average annual income rose to $23,286 by the end of 1999.

Exit Ticket Task

Navigate to our MAT142 Exit Ticket Form, answer the questions, and complete the *task* below.

Note. Today's discussion is listed as `8. Linear Functions and Equations`

Task: Find the equation of the line passing through $(2, -1)$ and perpendicular to $y = \frac{1}{4}x + 3$. Write your answer in both point-slope and slope-intercept forms.

Summary and Next Time...

Ideas From Today

- A **linear function** has the form $f(x) = mx + b$; slope m is the rate of change, $(0, b)$ is the y-intercept.
- **Slope-intercept form** $y = mx + b$ is easiest for reading; **point-slope form** $y - y_0 = m(x - x_0)$ is easiest for constructing.
- Point-slope form *is* the transformations framework applied to $f(x) = x$.
- Parallel lines have equal slopes; perpendicular lines have slopes with product -1.
- Solving a linear equation is equivalent to finding the intersection of two lines or the root of a function.
- Each algebraic step changes the form but preserves the solution location.

Resources

- Linear functions (Khan Academy)

Next Time:
Polynomial Functions
Homework:
Complete Homework 6 on MyOpenMath