A Sample Binomial Distribution Problem

Background

Remark 1 (Binomial Distribution) *The Binomial Distribution can be used in scenarios where*

- *We have some number, n, of repeated trials*
- *Each trial has two possible outcomes, which we can define as success and failure.*
- *The probability of success remains constant, regardless of the outcomes of previous trials. That is, the trials are independent of one another.*
- *We are interested in the probability of observing some number(s) of successes, where we are not concerned with which trials those successes appear on.*

Definition 1 (Binomial Distribution) *The binomial distribution gives us a method for computing the probability of observing k successes out of a set of n trials for which the probability of success on a single trial is p. Let X denote the number of successes observed within the n trials. Mathematically, the binomial distribution is described by*

$$\mathbb{P}\left[X = k\right] = \frac{n!}{k!\,(n-k)!} \cdot p^k \cdot (1-p)^{n-k}$$

This formula can be explained as follows:

- $\dfrac{n!}{k!\,(n-k)!}$ *gives the number of ways that k successes can be distributed among the n trials.*
- p^k *gives the probability of observing k successes.*
- $(1-p)^{n-k}$ *gives the probability associated with observing $(n-k)$ failures – note that k successes and $(n-k)$ failures encompasses the n total trials.*

In `R` we have the following two functions which evaluate the binomial distribution so that we do not need to do it "by hand":

- $\boxed{\mathbb{P}\left[X = k\right] \approx \texttt{dbinom(k, n, p)}}$
  - *For example, the probability of observing exactly four heads on seven flips of a fair coin is given by* `dbinom(4, 7, 0.5)` ≈ 0.2734.
- $\boxed{\mathbb{P}\left[X \leq k\right] \approx \texttt{pbinom(k, n, p)}}$
  - *For example, the probability of observing at most five heads on seven flips of a fair coin is given by* `pbinom(5, 7, 0.5)` ≈ 0.9375.

Example

Example 1 (Game of dreidel) *A dreidel is a four-sided spinning top with the Hebrew letters nun, gimel, hei, and shin, one on each side. Each side is equally likely to come up in a single spin of the dreidel. Suppose you spin a dreidel five times. Calculate the probability of getting*

a) exactly 2 nuns?
Solution. *Notice that spinning a dreidel five times fits the description required for using the binomial distribution. Each spin is one trial, we can argue that the two outcomes relevant to this particular part of the question are (i) nun, and (ii) non-nun. The probability of spinning a nun is $1/4$ since the dreidel has four sides and each side is equally likely. Furthermore, the probability of observing a nun does not*

change given the results of prior dreidel spins.
Now notice that we are interested in the probability of observing exactly two nuns out of five spins. From above, we see that R*'s* `dbinom()` *function is appropriate here. Let X denote the number of nuns spun during the five dreidel spins.*

$$X: \; 0 \;\; 1 \;\; \boxed{2} \; 3 \;\; 4 \;\; 5$$

$$\mathbb{P}[X = 2] = \texttt{dbinom}(2, 5, 0.25) \approx \boxed{0.2637}$$

b) *exactly 1 hei?*
Solution. Again we can see that the binomial distribution fits this scenario. Here the two outcomes are (i) hei, and (ii) non-hei. Again, the probability of observing a hei is $1/4$. Let X denote the number of heis spun out of the five dreidel spins.

$$X: \; 0 \;\; \boxed{1} \; 2 \;\; 3 \;\; 4 \;\; 5$$

$$\mathbb{P}[X = 1] = \texttt{dbinom}(1, 5, 0.25) \approx \boxed{0.3955}$$

c) *at most 2 gimels?*
Solution. The binomial distribution is a fit again, but notice that now we are interested in more than one possibility. Instead of exactly some number of successes we are now interested in at-most some number of successes (gimels). Let X denote the number of gimels spun out of the five dreidel spins.

$$X: \; \boxed{0 \;\; 1 \;\; 2} \; 3 \;\; 4 \;\; 5$$

$$\mathbb{P}[X \leq 2] = \texttt{pbinom}(2, 5, 0.25) \approx \boxed{0.8965}$$

d) *at least two nuns?*
Solution. As with each of the previous parts of this problem we can use the binomial distribution. Instead of exactly some number of successes or at-most some number of successes we are now interested in at-least some number of successes (nunss). Let X denote the number of nunss spun out of the five dreidel spins.

$$X: \; 0 \;\; 1 \; \boxed{2 \;\; 3 \;\; 4 \;\; 5}$$

Unlike the previous parts to this problem, however, neither of our binomial functions can be used to directly compute this probability. The `pbinom()` *function computes probability from the left (that is, starting with 0 successes). Instead of computing our desired probability directly, we begin with all of the events (a probability of 1, or 100%) and remove the probability corresponding to the events we are excluding – these events are in the red box below.*

$$X: \; \boxed{0 \;\; 1} \; \boxed{2 \;\; 3 \;\; 4 \;\; 5}$$

$$\mathbb{P}[X \geq 2] = 1 - \texttt{pbinom}(1, 5, 0.25) \approx 1 - 0.6328 = \boxed{0.3672}$$

e) *more than one, but at most four nuns?*
Solution. We'll make use the binomial distribution again. Because of the complexity of this event, a picture is nearly necessary if we are going to organize a strategy and calculate the correct probability.

$$X: \; 0 \;\; 1 \; \boxed{2 \;\; 3 \;\; 4} \; 5$$

Again, we are in a scenario where neither of our binomial functions can be used to directly compute this probability. It is worth reiterating that the `pbinom()` *function computes probability from the left (that is, starting with 0 successes), but the outcome we are interested in does not include 0 nuns. We also can't use the exact strategy we utilized previously (subtracting from 1) because removing the unwanted events "from the left" still leaves us with unwanted events "on the right". We could, however, use a similar* *strategy of starting with a probability that we know is "too big" and then removing the probabilities of the events we aren't interested in.*

$$X : \boxed{\boxed{0 \;\; 1} \; \boxed{2 \;\; 3 \;\; 4}} \; 5$$

$$\mathbb{P}\left[1 < X \leq 4\right] = \texttt{pbinom}\left(4, 5, 0.25\right) - \texttt{pbinom}\left(1, 5, 0.25\right) \approx \boxed{0.3662}$$

f) Find the probability that the first nun spun by a child is on their third spin.
Solution. *Unlike the previous parts to this problem, part (e) does not fit the assumptions of the binomial distribution. We are now in a scenario where we do care which spin leads to which outcome; the binomial distribution can not be used any longer. We proceed by reasoning out how to compute the probability.*
Notice that we still have five spins of the dreidel. We know that on the first two spins there should be no nun – the probability of a non-nun is $3/4$ on each of these spins. On the third spin we should observe a nun, which occurs with a probability of $1/4$. On the final two spins, any outcomes are acceptable – that is four out of the four possible outcomes are acceptable on each of these spins. Thus, we compute the probability as follows:

$$(3/4) \cdot (3/4) \cdot (1/4) \cdot (4/4) \cdot (4/4) \approx \boxed{0.1406}$$