MAT 350: Introduction to Eigenvectors and Eigenvalues

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Warm-Up Problems

Complete the following warm-up problems to re-familiarize yourself with concepts we'll be leveraging today.

Example: Consider the matrix $A = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}$. Evaluate the following matrix-vector products.

i. $A \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

ii. $A \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

iii. $A \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

iv. $A \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Look closely at the matrix-vector products and determine if there is anything particularly interesting about two of them.

Reminders and Today's Goal

- Multiplying an $n \times 1$ vector $\vec{x}$ by an $m \times n$ matrix transforms the vector from $\mathbb{R}^n$ to $\mathbb{R}^m$
  - If the matrix is an $n \times n$ matrix, then the vector is moved within $\mathbb{R}^n$
  - Matrix-vector multiplication with a square matrix max perform *rotations*, *reflections*, *vertical shears*, *horizontal shears*, *stretches*, *compressions*, or a combination of those transformatons.

Goals for Today: After today's discussion, you should be able to

- define what is meant by an *eigenvector*
- describe what is meant by an *eigenvalue*
- articulate the connection between an *eigenvector* and its *eigenvalue*
- use properties of *eigenvalues* and *eigenvectors* to quickly perform matrix-vector multiplication

Motivating Utility of Eigenvectors and Eigenvalues

We'll motivate why we might care about *eigenvectors* and *eigenvalues* even before defining them! We'll use an example to do it.

Example: Consider the matrix $A = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}$ and the vectors $\overrightarrow{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ from the warm-up problems.

- If you didn't already do so, compute $A\overrightarrow{v_1}$.
- Similarly, compute $A\overrightarrow{v_2}$ if you haven't already.
- Compute $A\left(5\overrightarrow{v_1}\right)$
- Compute $A\left(\overrightarrow{v_1} + \overrightarrow{v_2}\right)$
- Compute $A\left(3\overrightarrow{v_1} - 6\overrightarrow{v_2}\right)$

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- Compute $A^2\overrightarrow{v_1}$
- Compute $A^5\overrightarrow{v_2}$
- Notice that $\left\{\overrightarrow{v_1},\ \overrightarrow{v_2}\right\}$ is a basis for $\mathbb{R}^2$ and that the vector $\vec{u} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$ can be written as $\vec{u} = 7\overrightarrow{v_1} + 2\overrightarrow{v_2}$. Compute $A\vec{u}$.

Eigenvalues and Eigenvectors

Definition (Eigenvector and Eigenvalue): An *eigenvector* of an $n \times n$ matrix A is a *non-zero* vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ.

- A scalar λ is called an *eigenvalue* of A if there is a non-trivial solution to the matrix equation $A\vec{x} = \lambda\vec{x}$.

Geometrical Note: An *eigenvector* of the matrix A is a vector $\vec{x}$ whose transformed position after left-multiplication by the matrix A is just a scaling of the vector from its original position.

- These are vectors whose directionality has remained fixed (or have been rotated by 180°).

Identifying Eigenvectors and Eigenvalues

Strategy: Given a matrix A and a *non-zero* vector $\vec{v}$, the vector is an *eigenvector* of A if $A\vec{x} = \lambda\vec{x}$. That is, the matrix multiplication simply scales the original vector.

Example: Determine whether either of the vectors $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ or $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an *eigenvector* of the matrix $A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. If either is an *eigenvector*, identify its corresponding *eigenvalue*.

Solution Path. To answer this, do out the multiplication $A\vec{v}$ and $A\vec{u}$. Notice that $A\vec{v} = 4\vec{v}$, but $A\vec{u}$ is not simply a scaled copy of $\vec{u}$. The *eigenvalue* corresponding to the *eigenvector* $\vec{v}$ is $\lambda = 4$.

Finding Eigenvectors and Eigenvalues

This will be the focus of our next discussion, but for now, a preview…

Strategy (Finding Eigenvectors and Eigenvalues): Recall that a scalar λ is an *eigenvalue* of the matrix A if there exists a *non-trivial* (not $\vec{0}$) solution to the matrix equation $A\vec{x} = \lambda\vec{x}$.

Intuition: We haven't solved equations with unknowns on both sides of an equal sign this semester, so we'll begin by rewriting the equation

$$A\vec{x} = \lambda\vec{x}$$

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$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ \implies \quad A\vec{x} - \lambda\vec{x} &= \vec{0} \end{aligned}$$

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Intuition: We haven't solved equations with unknowns on both sides of an equal sign this semester, so we'll begin by rewriting the equation

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ \implies A\vec{x} - \lambda\vec{x} &= \vec{0} \\ \implies (A - \lambda I)\,\vec{x} &= \vec{0} \end{aligned}$$

The bottom equation will have a *unique* solution if $(A - \lambda I)$ is *invertible*.

We'll have *non-trivial* solutions if $(A - \lambda I)$ is ***not*** *invertible*.

Finding Eigenvectors and Eigenvalues

Strategy (Finding Eigenvectors and Eigenvalues): Recall that a scalar λ is an *eigenvalue* of the matrix A if there exists a *non-trivial* (not $\vec{0}$) solution to the matrix equation $A\vec{x} = \lambda\vec{x}$.

Note: The following items follow directly from what we just saw and are worth noting regarding eigenvalues.

- Notice that λ is an *eigenvalue* of A if $(A - I_n)\,\vec{x} = \vec{0}$ has a non-trivial solution.
- Notice that λ is an *eigenvalue* of A if $\left[\begin{array}{c|c} A - \lambda I_n & \vec{0} \end{array}\right]$ has a free variable.
- Notice that λ is an *eigenvalue* of A if the matrix $A - \lambda I_n$ is not invertible.
- Notice that λ is an *eigenvalue* of A if $\dim(\mathrm{Nul}\,(A - \lambda I_n)) > 0$.
- Notice that, if λ is an *eigenvalue* of A, then $\mathrm{Nul}\,(A - \lambda I_n)$ is a subspace of $\mathbb{R}^n$ corresponding to the eigenvalue λ. This subspace is often referred to as the *eigenspace* of A corresponding to λ.

Completed Example #2

Example: Determine whether $\lambda = 5$ is an eigenvalue for the matrix $A = \begin{bmatrix} 6 & 8 \\ 1 & 13 \end{bmatrix}$. If it is an eigenvalue, find an eigenvector corresponding to $\lambda = 5$.

We'll start by solving the matrix equation $(A - 5I)\vec{x} = \vec{0}$.

As usual, we'll do this by constructing a corresponding augmented matrix and row-reducing.

$$\left[\begin{array}{cc|c} 6-5 & 8 & 0 \\ 1 & 13-5 & 0 \end{array}\right]$$

```
import sympy as sp

A = sp.Matrix([[6 - 5, 8, 0], [1, 13 -
A.rref()
```

```
(Matrix([
[1, 8, 0],
[0, 0, 0]]), (0,))
```

There is a free variable here, so $\lambda = 5$ is indeed a eigenvalue for this matrix.

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[0, 0, 0]]), (0,))
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There is a free variable here, so $\lambda = 5$ is indeed a eigenvalue for this matrix.

We can construct a basis for the eigenspace of this matrix corresponding to $\lambda = 5$ by writing the solutions to the equation we began from, in parameteric vector form.

Note that $\vec{x} = x_2 \begin{bmatrix} -8 \\ 1 \end{bmatrix}$. Thus, $\mathcal{B}_{\lambda=5} = \left\{ \begin{bmatrix} -8 \\ 1 \end{bmatrix} \right\}$.

Eigenvalues of Triangular Matrices

Recall (Triangular Matrix): A matrix A having all entries either above or below its main diagonal as 0′s is called a *triangular* matrix. If the 0′s are below the main diagonal, A is called *lower triangular* while a matrix having all 0′s above the main diagonal is $upper triangular*.

Theorem (Eigenvalues of Triangular Matrices): The eigenvalues of a triangular matrix are the entries along its main diagonal.

Example: The *eigenvalues* of the matrix $A = \begin{bmatrix} 6 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & 0 & -7 & 5 \end{bmatrix}$ are…

Eigenvalues of Triangular Matrices

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Theorem (Eigenvalues of Triangular Matrices): The eigenvalues of a triangular matrix are the entries along its main diagonal.

Example: The *eigenvalues* of the matrix $A = \begin{bmatrix} 6 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & 0 & -7 & 5 \end{bmatrix}$ are $\lambda_1 = 6$, $\lambda_2 = 0$, $\lambda_3 = 1$, and $\lambda_4 = 5$.

Theorem: If $\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ of an $n \times n$ matrix A then the set $\left\{\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_r}\right\}$ are linearly independent.

Eigenvectors and Eigenvalues, Why Care?

- Scalar multiplication is much simpler and faster than matrix multiplication.
- Eigenvectors allow us to replace matrix multiplication by scalar multiplication.
- Recall the *superposition principle* of *linear transformations* which is often utilized in physics and engineering

$$T\left(c_1\overrightarrow{v_1}+c_2\overrightarrow{v_2}+\cdots+c_n\overrightarrow{v_n}\right)=c_1T\left(\overrightarrow{v_1}\right)+c_2T\left(\overrightarrow{v_2}\right)+\cdots+c_nT\left(\overrightarrow{v_n}\right)$$

- This is particularly useful when $\left\{\overrightarrow{v_1},\overrightarrow{v_2},\cdots,\overrightarrow{v_n}\right\}$ form a *basis* for $\mathbb{R}^n$.
- Even better, if we had a basis consisting of *eigenvectors* (an *eigenbasis*), then there is no need for matrix multiplication to be carried out at all.
  - Write a generic vector $\vec{x}$ as a *linear combination* of the *eigenbasis* vectors and then evaluate $T\left(\vec{x}\right)$ using scalar multiplication and addition.

Eigenvectors and Eigenvalues, Why Care?

- Additionally, *eigenvectors* and *eigenvalues* indicate how much (*eigenvalues*) and in what directions (*eigenvectors*) a linear transformation stretches space.
- We've seen that changing *bases* from the *standard basis* to an alternative basis (like an *eigenbasis*) can be helpful.
- If one or more of the *eigenvectors* (axes in the *eigenbasis* representation of space) have very small *eigenvalues*, then…
  - those extra spatial dimensions may not be necessary
  - we may be able to drop them without much information loss

Examples to Try

Example 1: Determine whether $\lambda = 3$ is an eigenvalue for the matrix $A = \begin{bmatrix} 5 & 6 \\ -2 & 4 \end{bmatrix}$. If it is an eigenvalue, find a corresponding eigenvector.

Example 2: Determine whether the vector $\begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$ is an eigenvector for the matrix $A = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix}$. If so, find the corresponding eigenvalue and at least one other eigenvector corresponding to the same eigenvalue.

Examples to Try (2 of 2)

Example 3: The matrix $A = \begin{bmatrix} 8 & 2 \\ 6 & 12 \end{bmatrix}$ has eigenvalues $\lambda = 6$ and $\lambda = 14$. Find a basis for each of the corresponding eigenspaces.

Example 4: Find the eigenvalues corresponding to the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

Homework

Start Homework 11
on MyOpenMath

Next Time…

Finding
Eigenvectors and Eigenvalues